State-constrained Linear-Quadratic Optimal Control is a shape-constrained kernel regression

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Where to use Machine Learning in control theory?

Many objects can be learnt depending on the available data

- Trajectory $x : t \in [t_0, T] \mapsto \mathbb{R}^Q$
- Control $u : t \in [t_0, T] \mapsto \mathbb{R}^P$
- Vector field $f : (t, x, u) \mapsto \mathbb{R}^Q$
- Lagrangian $L : (t, x, u) \mapsto \mathbb{R} \cup \{\infty\}$
- Value function $V_{T,x_T} : (t_0, x_0) \mapsto \mathbb{R} \cup \{\infty\}$

Which one should we try to approximate?

What is the most principled/theoretically grounded application of kernel methods?

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What is the most principled/theoretically grounded application of kernel methods?

Trajectories of linear systems belong to a reproducing kernel Hilbert space (RKHS)!

State constraints are then easy to satisfy!

Table of Contents

This talk summarizes

- *Linearly-constrained Linear Quadratic Regulator from the viewpoint of kernel methods*, Aubin, **SIAM J. on Control and Optimization, 2021**
- *Interpreting the dual Riccati equation through the LQ reproducing kernel*, Aubin, **Comptes Rendus. Mathématique, 2021**

The code is available at `https://github.com/PCAubin`.

Follow-ups are on their way

Operator-valued Kernels and Control of Infinite dimensional Dynamic Systems, Aubin, Alain Bensoussan **arxiv.org/abs/2206.09419**

Time-varying state-constrained LQ optimal control

$$\min_{\mathbf{x}(\cdot),\mathbf{u}(\cdot)} \quad \chi_{\mathbf{x}_0}(\mathbf{x}(t_0)) + g(\mathbf{x}(T))$$

$$+\mathbf{x}(t_{ref})^\top \mathbf{J}_{ref}\mathbf{x}(t_{ref}) + \int_{t_0}^{T} \left[\mathbf{x}(t)^\top \mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}(t)^\top \mathbf{R}(t)\mathbf{u}(t)\right] \mathrm{d}t$$

$$\text{s.t.} \quad \mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \text{ a.e. in } [t_0, T],$$

$$\mathbf{c}_i(t)^\top \mathbf{x}(t) \leq d_i(t), \ \forall\, t \in \mathcal{T}_c, \forall\, i \in [\mathcal{I}] = [\![1, \mathcal{I}]\!],$$

- state $\mathbf{x}(t) \in \mathbb{R}^Q$, control $\mathbf{u}(t) \in \mathbb{R}^P$,
- reference time $t_{ref} \in [t_0, T]$, set of constraint times $\mathcal{T}_c \subset [t_0, T]$,
- $\mathbf{A}(\cdot) \in L^1(t_0, T)$, $\mathbf{B}(\cdot) \in L^2(t_0, T)$, $\mathbf{Q}(\cdot) \in L^1(t_0, T)$, $\mathbf{R}(\cdot) \in L^2(t_0, T)$,
- $\mathbf{Q}(t) \succcurlyeq 0$ and $\mathbf{R}(t) \succcurlyeq r\mathrm{Id}_M$ $(r > 0)$, $\mathbf{c}_i(\cdot), d_i(\cdot) \in C^0(t_0, T)$, $\mathbf{J}_{ref} \succ \mathbf{0}$,
- lower-semicontinuous terminal cost $g : \mathbb{R}^Q \to R \cup \{\infty\}$, indicator function $\chi_{\mathbf{x}_0}$,

- $\mathbf{x}(\cdot) : [t_0, T] \to \mathbb{R}^Q$ absolutely continuous, $\mathbf{R}(\cdot)^{1/2}\mathbf{u}(\cdot) \in L^2([t_0, T])$

Time-varying state-constrained LQ optimal control

$$\min_{\mathbf{x}(\cdot),\mathbf{u}(\cdot)} \quad \chi_{\mathbf{x}_0}(\mathbf{x}(t_0)) + g(\mathbf{x}(T)) \qquad \rightarrow L(\mathbf{x}(t_j)_{j\in[J]})$$

$$+\mathbf{x}(t_{ref})^\top \mathbf{J}_{ref}\mathbf{x}(t_{ref}) + \int_{t_0}^{T} \left[\mathbf{x}(t)^\top \mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}(t)^\top \mathbf{R}(t)\mathbf{u}(t)\right] \mathrm{d}t \qquad \rightarrow \|\mathbf{x}(\cdot)\|^2_{\mathcal{S}}$$

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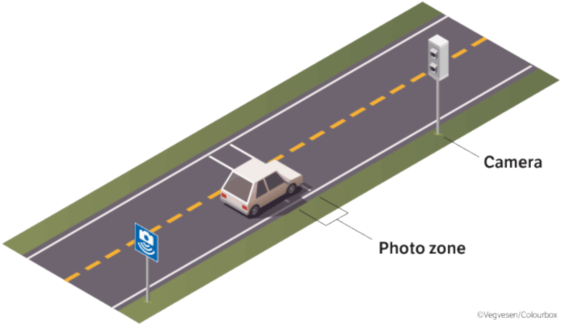
- lower-semicontinuous terminal cost $g : \mathbb{R}^Q \rightarrow R \cup \{\infty\}$, indicator function $\chi_{\mathbf{x}_0}$, "loss function" $L : (\mathbb{R}^Q)^J \rightarrow \mathbb{R} \cup \{\infty\}$,
- $\mathbf{x}(\cdot) : [t_0, T] \rightarrow \mathbb{R}^Q$ absolutely continuous, $\mathbf{R}(\cdot)^{1/2}\mathbf{u}(\cdot) \in L^2([t_0, T])$

Why are state constraints difficult to study?

- **Theoretical obstacle**: Pontryagine's Maximum Principle involves not only an adjoint vector $\mathbf{p}(t)$ but also measures/BV functions $\psi(t)$ supported at times where the constraints are saturated. You cannot just backpropagate the Hamiltonian system from the transversality condition.

- **Numerical obstacle**: Time discretization of constraints may fail e.g.



Speed cameras in traffic control

In between two cameras, drivers always break the speed limit.

Reproducing kernel Hilbert spaces (RKHS)

A RKHS $(\mathcal{F}_k, \langle \cdot, \cdot \rangle_{\mathcal{F}_k})$ is a Hilbert space of real-valued functions over a set $\mathcal{T}$ if one of the following equivalent conditions is satisfied [Aronszajn, 1950]

$\exists\, k : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ s.t. $k_t(\cdot) = k(\cdot, t) \in \mathcal{F}_k$ and $f(t) = \langle f(\cdot), k_t(\cdot) \rangle_{\mathcal{F}_k}$ for all $t \in \mathcal{T}$ and $f \in \mathcal{F}_k$ (reproducing property)

the topology of $(\mathcal{F}_k, \langle \cdot, \cdot \rangle_{\mathcal{F}_k})$ is stronger than pointwise convergence
i.e. $\delta_t : f \in \mathcal{F}_k \mapsto f(t)$ is continuous for all $t \in \mathcal{T}$.

$$|f(t) - f_n(t)| = |\langle f - f_n, k_t \rangle_{\mathcal{F}_k}| \leq \|f - f_n\|_{\mathcal{F}_k} \|k_t\|_{\mathcal{F}_k} = \|f - f_n\|_{\mathcal{F}_k} \sqrt{k(t,t)}$$

For $\mathcal{T} \subset \mathbb{R}^d$, Sobolev spaces $\mathcal{H}^s(\mathcal{T}, \mathbb{R})$ satisfying $s > d/2$ are RKHSs.

$$\begin{cases} H_0^1 = \{f \mid f(0) = 0,\ \exists f' \in L^2(0, \infty)\} \\ \langle f, g \rangle_{H_0^1} = \int_0^\infty f' g' \mathrm{d}t \end{cases} \longleftrightarrow k(t,s) = \min(t,s).$$

Other classical kernels

$$k_{\text{Gauss}}(t,s) = \exp\left(-\|t - s\|_{\mathbb{R}^d}^2 / (2\sigma^2)\right) \quad k_{\text{poly}}(t,s) = (1 + \langle t, s \rangle_{\mathbb{R}^d})^2.$$

Two essential tools for computations

Representer Theorem (e.g. [Schölkopf et al., 2001])

Let $L : \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$, strictly increasing $\Omega : \mathbb{R}_+ \to \mathbb{R}$, and

$$\bar{f} \in \arg\min_{f \in \mathcal{F}_k} L\left(\left(f(t_n)\right)_{n\in[N]}\right) + \Omega\left(\|f\|_k\right)$$

Then $\exists\, (a_n)_{n\in[N]} \in \mathbb{R}^N$ s.t. $\bar{f}(\cdot) = \sum_{n\in[N]} a_n k(\cdot, t_n)$

$\hookrightarrow$ Optimal solutions lie in a finite dimensional subspace of $\mathcal{F}_k$.

Finite number of evaluations $\implies$ finite number of coefficients

Kernel trick

$$\langle \sum_{n\in[N]} a_n k(\cdot, t_n), \sum_{m\in[M]} b_m k(\cdot, s_m)\rangle_{\mathcal{F}_k} = \sum_{n\in[N]} \sum_{m\in[M]} a_n b_m k(t_n, s_m)$$

$\hookrightarrow$ On this finite dimensional subspace, no need to know $(\mathcal{F}_k, \langle\cdot,\cdot\rangle_{\mathcal{F}_k})$.

Vector-valued reproducing kernel Hilbert space (vRKHS)

Definition (vRKHS)

Let $\mathcal{T}$ be a non-empty set. A Hilbert space $(\mathcal{F}_K, \langle \cdot, \cdot \rangle_K)$ of $\mathbb{R}^Q$-vector-valued functions defined on $\mathcal{T}$ is a vRKHS if there exists a matrix-valued kernel $K : \mathcal{T} \times \mathcal{T} \to \mathbb{R}^{Q \times Q}$ such that the reproducing property holds:

$$K(\cdot, t)\mathbf{p} \in \mathcal{F}_K, \quad \mathbf{p}^\top \mathbf{f}(t) = \langle \mathbf{f}, K(\cdot, t)\mathbf{p} \rangle_K, \quad \text{for } t \in \mathcal{T}, \mathbf{p} \in \mathbb{R}^Q, \mathbf{f} \in \mathcal{F}_K$$

There is a one-to-one correspondence between K and $(\mathcal{F}_K, \langle \cdot, \cdot \rangle_K)$ [Micheli and Glaunès, 2014], so changing $\mathcal{T}$ or $\langle \cdot, \cdot \rangle_K$ changes K.

Representer theorem in vRKHSs

Theorem (Representer theorem with constraints, P.-C. Aubin, 2021)

Let $(\mathcal{F}_K, \langle \cdot, \cdot \rangle_K)$ be a vRKHS defined on a set $\mathcal{T}$. For a "loss" $L : \mathbb{R}^{N_0} \to \mathbb{R} \cup \{+\infty\}$, strictly increasing "regularizer" $\Omega : \mathbb{R}_+ \to \mathbb{R}$, and constraints $d_i : \mathbb{R}^{N_i} \to \mathbb{R}$, consider the optimization problem

$$\bar{\mathbf{f}} \in \arg\min_{\mathbf{f} \in \mathcal{F}_K} \quad L\left(\mathbf{c}_{0,1}^\top \mathbf{f}(t_{0,1}), \ldots, \mathbf{c}_{0,N_0}^\top \mathbf{f}(t_{0,N_0})\right) + \Omega\left(\|\mathbf{f}\|_K\right)$$

$$\text{s.t.}$$

$$\lambda_i \|\mathbf{f}\|_K \leq d_i(\mathbf{c}_{i,1}^\top \mathbf{f}(t_{i,1}), \ldots, \mathbf{c}_{i,N_i}^\top \mathbf{f}(t_{i,N_i})), \forall\, i \in [\![1, P]\!].$$

Then there exists $\{\mathbf{p}_{i,m}\}_{m \in [\![1,N_i]\!]} \subset \mathbb{R}^Q$ and $\alpha_{i,m} \in \mathbb{R}$ such that

$$\bar{\mathbf{f}} = \sum_{i=0}^{P} \sum_{m=1}^{N_i} K(\cdot, t_{i,m})\mathbf{p}_{i,m} \text{ with } \mathbf{p}_{i,m} = \alpha_{i,m}\mathbf{c}_{i,m}.$$

Objective: Turn the state-constrained LQR into “KRR“

We have a vector space $\mathcal{S}$ of controlled trajectories $\mathbf{x}(\cdot) : [t_0, T] \to \mathbb{R}^Q$

$$\mathcal{S}_{[t_0,T]} := \{\mathbf{x}(\cdot) \,|\, \exists\, \mathbf{u}(\cdot) \in L^2(t_0, T) \text{ s.t. } \mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \text{ a.e. } \}$$

Given $\mathbf{x}(\cdot) \in \mathcal{S}_{[t_0,T]}$, for the pseudoinverse $\mathbf{B}(t)^{\ominus}$ of $\mathbf{B}(t)$, set

$$\mathbf{u}(t) := \mathbf{B}(t)^{\ominus}[\mathbf{x}'(t) - \mathbf{A}(t)\mathbf{x}(t)] \text{ a.e. in } [t_0, T].$$

$$\begin{aligned}\langle \mathbf{x}_1(\cdot), \mathbf{x}_2(\cdot)\rangle_{\mathcal{S}} &:= \mathbf{x}_1(t_{ref})^\top \mathbf{J}_{ref}\mathbf{x}_2(t_{ref}) \\ &+ \int_{t_0}^{T} \left[\mathbf{x}_1(t)^\top \mathbf{Q}(t)\mathbf{x}_2(t) + \mathbf{u}_1(t)^\top \mathbf{R}(t)\mathbf{u}_2(t)\right] \mathrm{d}t\end{aligned}$$

LQR for $\mathbf{Q} \equiv \mathbf{0}$, $\mathbf{R} \equiv \mathrm{Id}$

$$\min_{\substack{\mathbf{x}(\cdot)\in\mathcal{S} \\ \mathbf{u}(\cdot)\in L^2}} L(\mathbf{x}(t_j)_{j\in[J]}) + \|\mathbf{u}(\cdot)\|^2_{L^2(t_0,T)}$$

$$\mathbf{c}_i(t)^\top \mathbf{x}(t) \le d_i(t), t \in \mathcal{T}_c, i \in [\mathcal{I}]$$

“KRR“ (Kernel Ridge Regression)

$$\min_{\mathbf{x}(\cdot)\in\mathcal{S}} L(\mathbf{x}(t_j)_{j\in[J]}) + \|\mathbf{x}(\cdot)\|^2_{\mathcal{S}}$$

$$\mathbf{c}_i(t)^\top \mathbf{x}(t) \le d_i(t), t \in \mathcal{T}_c, i \in [\mathcal{I}]$$

Is $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$ a RKHS?

Objective: Turn the state-constrained LQR into "KRR"

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$$\begin{aligned} \mathbf{u}(t) &:= \mathbf{B}(t)^{\ominus}[\mathbf{x}'(t) - \mathbf{A}(t)\mathbf{x}(t)] \text{ a.e. in } [t_0, T]. \\ \langle \mathbf{x}_1(\cdot), \mathbf{x}_2(\cdot)\rangle_{\mathcal{S}} &:= \mathbf{x}_1(t_{ref})^\top \mathbf{J}_{ref}\mathbf{x}_2(t_{ref}) \\ &\quad + \int_{t_0}^{T} \left[\mathbf{x}_1(t)^\top \mathbf{Q}(t)\mathbf{x}_2(t) + \mathbf{u}_1(t)^\top \mathbf{R}(t)\mathbf{u}_2(t)\right] \mathrm{d}t \end{aligned}$$

Lemma (P.-C. Aubin, SICON 2021)

$(\mathcal{S}_{[t_0,T]}, \langle\cdot,\cdot\rangle_{\mathcal{S}})$ *is a vRKHS over* $[t_0, T]$ *with uniformly continuous* $K(\cdot,\cdot;[t_0,T])$.

Splitting $\mathcal{S}_{[t_0,T]}$ into subspaces and identifying their kernels

It is hard to identify K, but take $\mathbf{Q} \equiv \mathbf{0}$, $\mathbf{R} \equiv \mathrm{Id}$, $t_{ref} = t_0$, $\mathbf{J}_{ref} = \mathrm{Id}$

$$\langle \mathbf{x}_1(\cdot), \mathbf{x}_2(\cdot) \rangle_{\mathcal{S}} := \mathbf{x}_1(t_0)^\top \mathbf{x}_2(t_0) + \int_{t_0}^{T} \mathbf{u}_1(t)^\top \mathbf{u}_2(t) \mathrm{d}t.$$

$$\mathcal{S}_0 := \{\mathbf{x}(\cdot) \,|\, \mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t), \text{ a.e. in } [t_0, T]\} \qquad \|\mathbf{x}(\cdot)\|_{K_0}^2 = \|\mathbf{x}(t_0)\|^2$$

$$\mathcal{S}_u := \{\mathbf{x}(\cdot) \,|\, \mathbf{x}(\cdot) \in \mathcal{S} \text{ and } \mathbf{x}(t_0) = 0\} \qquad \|\mathbf{x}(\cdot)\|_{K_1}^2 = \|\mathbf{u}(\cdot)\|_{L^2(t_0,T)}^2.$$

As $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_u$, $K = K_0 + K_1$.

Splitting $\mathcal{S}_{[t_0,T]}$ into subspaces and identifying their kernels

It is hard to identify K, but take $\mathbf{Q} \equiv \mathbf{0}$, $\mathbf{R} \equiv \mathrm{Id}$, $t_{ref} = t_0$, $\mathbf{J}_{ref} = \mathrm{Id}$

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As $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_u$, $K = K_0 + K_1$. Since $\dim(\mathcal{S}_0) = Q$, for $\mathbf{\Phi}_{\mathbf{A}}(t,s) \in \mathbb{R}^{Q \times Q}$ the state-transition matrix $s \to t$ of $\mathbf{x}'(\tau) = \mathbf{A}(\tau)\mathbf{x}(\tau)$

$$K_0(s,t) = \mathbf{\Phi}_{\mathbf{A}}(s,t_0)\mathbf{\Phi}_{\mathbf{A}}(t,t_0)^\top.$$

Splitting $\mathcal{S}_{[t_0,T]}$ into subspaces and identifying their kernels

It is hard to identify K, but take $\mathbf{Q} \equiv \mathbf{0}$, $\mathbf{R} \equiv \mathrm{Id}$, $t_{ref} = t_0$, $\mathbf{J}_{ref} = \mathrm{Id}$

$$\langle \mathbf{x}_1(\cdot), \mathbf{x}_2(\cdot)\rangle_{\mathcal{S}} := \mathbf{x}_1(t_0)^\top \mathbf{x}_2(t_0) + \int_{t_0}^{T} \mathbf{u}_1(t)^\top \mathbf{u}_2(t)\mathrm{d}t.$$

$$\mathcal{S}_0 := \{\mathbf{x}(\cdot) \,|\, \mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t), \text{ a.e. in } [t_0, T]\} \qquad \|\mathbf{x}(\cdot)\|^2_{K_0} = \|\mathbf{x}(t_0)\|^2$$

$$\mathcal{S}_u := \{\mathbf{x}(\cdot) \,|\, \mathbf{x}(\cdot) \in \mathcal{S} \text{ and } \mathbf{x}(t_0) = 0\} \qquad \|\mathbf{x}(\cdot)\|^2_{K_1} = \|\mathbf{u}(\cdot)\|^2_{L^2(t_0,T)}.$$

As $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_u$, $K = K_0 + K_1$. Since $\dim(\mathcal{S}_0) = Q$, for $\mathbf{\Phi}_{\mathbf{A}}(t,s) \in \mathbb{R}^{Q \times Q}$ the state-transition matrix $s \to t$ of $\mathbf{x}'(\tau) = \mathbf{A}(\tau)\mathbf{x}(\tau)$

$$K_0(s,t) = \mathbf{\Phi}_{\mathbf{A}}(s,t_0)\mathbf{\Phi}_{\mathbf{A}}(t,t_0)^\top.$$

K_1 obtained using only the reproducing property and variation of constants

$$K_1(s,t) = \int_{t_0}^{\min(s,t)} \mathbf{\Phi}_{\mathbf{A}}(s,\tau)\mathbf{B}(\tau)\mathbf{B}(\tau)^\top \mathbf{\Phi}_{\mathbf{A}}(t,\tau)^\top \mathrm{d}\tau.$$

Examples: controllability Gramian/transversality condition

Steer a point from $(0,\mathbf{0})$ to $(T,\mathbf{x}_T)$, with e.g. $g(\mathbf{x}(T)) = \|\mathbf{x}_T - \mathbf{x}(T)\|_N^2$

Exact planning ($\mathbf{x}(T) = \mathbf{x}_T$)

$$\min_{\substack{\mathbf{x}(\cdot)\in\mathcal{S}\\ \mathbf{x}(0)=\mathbf{0}}} \quad \chi_{\mathbf{x}_T}(\mathbf{x}(T)) + \frac{1}{2}\|\mathbf{u}(\cdot)\|^2_{L^2(t_0,T)}$$

Relaxed planning ($\mathbf{g} \in \mathcal{C}^1$ convex)

$$\min_{\substack{\mathbf{x}(\cdot)\in\mathcal{S}\\ \mathbf{x}(0)=\mathbf{0}}} \quad g(\mathbf{x}(T)) + \frac{1}{2}\|\mathbf{u}(\cdot)\|^2_{L^2(t_0,T)}$$

$\mathbf{x}(0) = \mathbf{0} \Leftrightarrow \mathbf{x}(\cdot) \in \mathcal{S}_u$. Representer theorem: $\exists\, \mathbf{p}_T$, $\bar{\mathbf{x}}(\cdot) = K_1(\cdot, T)\mathbf{p}_T$

Controllability Gramian

$$K_1(T,T) = \int_0^T \mathbf{\Phi}_\mathbf{A}(T,\tau)\mathbf{B}(\tau)\mathbf{B}(\tau)^\top \mathbf{\Phi}_\mathbf{A}(T,\tau)^\top \mathrm{d}\tau$$

$\bar{\mathbf{x}}(T) = \mathbf{x}_T \Leftrightarrow \mathbf{x}_T \in \mathrm{Im}(K_1(T,T))$

Transversality Condition

$$\begin{aligned}\mathbf{0} &= \nabla\left(\mathbf{p} \mapsto g(K_1(T,T)\mathbf{p}) + \frac{1}{2}\mathbf{p}^\top K_1(T,T)\mathbf{p}\right)(\mathbf{p}_T)\\ &= K_1(T,T)(\nabla g(K_1(T,T)\mathbf{p}_T) + \mathbf{p}_T).\end{aligned}$$

Sufficient to take $\mathbf{p}_T = -\nabla g(\bar{\mathbf{x}}(T))$

Relation with the differential Riccati equation

Take $t_{ref} = T$, $\mathbf{J}_{ref} = \mathbf{J}_T \succ \mathbf{0}$. Let $J(t,T)$ be the solution of

$$\begin{aligned} -\partial_1 \mathbf{J}(t,T) &= \mathbf{A}(t)^\top \mathbf{J}(t,T) + \mathbf{J}(t,T)\mathbf{A}(t) \\ &\quad - \mathbf{J}(t,T)\mathbf{B}(t)\mathbf{R}(t)^{-1}\mathbf{B}(t)^\top \mathbf{J}(t,T) + \mathbf{Q}(t), \\ \mathbf{J}(T,T) &= \mathbf{J}_T, \end{aligned}$$

Theorem (P.-C. Aubin, 2021)

Let $K_{\text{diag}} : t_0 \in]-\infty, T] \mapsto K(t_0, t_0; [t_0, T])$. *Then* $K_{\text{diag}}(t_0) = \mathbf{J}(t_0, T)^{-1}$. *More generally,* $K(\cdot, t; [t_0, T])$ *is given by a matrix Hamiltonian system for all* $t \in [t_0, T]$

$$\begin{aligned} \partial_1 K(s,t) &= \mathbf{A}(s)K(s,t) + \mathbf{B}(s)\mathbf{R}(s)^{-1}\mathbf{B}(s)^\top \begin{cases} \mathbf{\Pi}(s,t) + \mathbf{\Phi}_\mathbf{A}(t_0,s)^\top - \mathbf{\Phi}_\mathbf{A}(t,s)^\top, s \geq t, \\ \mathbf{\Pi}(s,t) + \mathbf{\Phi}_\mathbf{A}(t_0,s)^\top, s < t. \end{cases} \\ \partial_1 \mathbf{\Pi}(s,t) &= -\mathbf{A}(s)^\top \mathbf{\Pi}(s,t) + \mathbf{Q}(s)K(s,t), \\ \mathbf{\Pi}(t_0,t) &= -Id_N, \\ K(t,T) &= -\mathbf{J}_T^{-1}(\mathbf{\Pi}(T,t)^\top + \mathbf{\Phi}_\mathbf{A}(t,T) - \mathbf{\Phi}_\mathbf{A}(t_0,T)). \end{aligned}$$

Relation with the differential Riccati equation

$$\bar{\mathbf{x}}(\cdot) := \underset{\mathbf{x}(\cdot)\in\mathcal{S}_{[t_0,T]}}{\arg\min} \quad \underbrace{\mathbf{x}(T)^\top \mathbf{J}_T\, \mathbf{x}(T) + \int_{t_0}^{T} [\mathbf{x}(t)^\top \mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}(t)^\top \mathbf{R}(t)\mathbf{u}(t)]\mathrm{d}t}_{\|\mathbf{x}(\cdot)\|^2_{\mathcal{S}}}$$

$$\text{s.t.}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0,$$

Pontryagine's Maximum Principle (PMP)

$\mathbf{p}(t) = -\mathbf{J}(t,T)\bar{\mathbf{x}}(t)$ and $\bar{\mathbf{u}}(t) = \mathbf{R}(t)^{-1}\mathbf{B}(t)^\top \mathbf{p}(t) = -\mathbf{R}(t)^{-1}\mathbf{B}(t)^\top \mathbf{J}(t,T)\bar{\mathbf{x}}(t) =: \mathbf{G}(t)\bar{\mathbf{x}}(t)$

$\hookrightarrow$ **online and differential** approach

Representer theorem from kernel methods

$\bar{\mathbf{x}}(t) = K(t, t_0; [t_0, T])\mathbf{p}_0$, with $\mathbf{p}_0 = K(t_0, t_0; [t_0, T])^{-1}\mathbf{x}_0 \in \mathbb{R}^Q$

$\hookrightarrow$ **offline and integral** approach ($\sim$ Green kernel in PDEs)

Numerical example: submarine in a cavern

Original control problem

$$\min_{z(\cdot)\in W^{2,2},u(\cdot)\in L^2} \int_0^1 |u(t)|^2 \mathrm{d}t$$

$$\text{s.t.}$$

$$z(0) = 0, \quad \dot{z}(0) = 0,$$

$$\ddot{z}(t) = -\dot{z}(t) + u(t), \ \forall t \in [0, 1],$$

$$z(t) \in [z_{\text{low}}(t), z_{\text{up}}(t)], \ \forall\, t \in [0, 1].$$

Numerical example: submarine in a cavern

Original control problem

$$\min_{z(\cdot)\in W^{2,2}, u(\cdot)\in L^2} \int_0^1 |u(t)|^2 \mathrm{d}t$$

$$\text{s.t.}$$

$$z(0) = 0, \quad \dot{z}(0) = 0,$$

$$\ddot{z}(t) = -\dot{z}(t) + u(t), \ \forall t \in [0,1],$$

$$z(t) \in [z_{\mathrm{low}}(t), z_{\mathrm{up}}(t)], \ \forall\, t \in [0,1].$$

Rewriting in standard form

$$\min_{\mathbf{x}(\cdot)\in W^{1,2}, u(\cdot)\in L^2} \int_0^1 |u(t)|^2 \mathrm{d}t$$

$$\text{s.t.}$$

$$\mathbf{x}(0) = 0,$$

$$\mathbf{x}'(t) \overset{\text{a.e.}}{=} \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$

$$z_1(t) \in [z_{\mathrm{low}}(t), z_{\mathrm{up}}(t)], \ \forall\, t \in [0,1]$$

$$\mathbf{x} = \begin{pmatrix} z \\ \dot{z} \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Numerical example: submarine in a cavern

RKHS regression

$$\min_{\mathbf{x}(\cdot) \in \mathcal{S}_u} \quad \|\mathbf{x}(\cdot)\|_{K_1}^2$$
$$\text{s.t.}$$
$$z_1(t) \in [z_{\text{low}}(t), z_{\text{up}}(t)],\ \forall\, t \in [0,1]$$

$$\mathcal{S}_u := \{\mathbf{x}(\cdot) \,|\, \mathbf{x}(\cdot) \in \mathcal{S} \text{ and } \mathbf{x}(0) = 0\}$$

$$\|\mathbf{x}(\cdot)\|_{K_1}^2 = \|\mathbf{u}(\cdot)\|_{L^2(0,1)}^2.$$

Rewriting in standard form

$$\min_{\mathbf{x}(\cdot)\in W^{1,2}, u(\cdot)\in L^2} \quad \int_0^1 |u(t)|^2 \mathrm{d}t$$
$$\text{s.t.}$$
$$\mathbf{x}(0) = 0,$$
$$\mathbf{x}'(t) \overset{\text{a.e.}}{=} \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$
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Numerical example: submarine in a cavern

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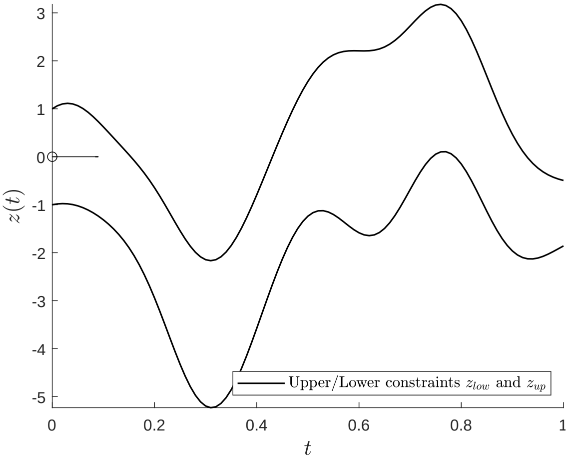
$$\|\mathbf{x}(\cdot)\|_{K_1}^2 = \|\mathbf{u}(\cdot)\|_{L^2(0,1)}^2.$$

$$K_1(s,t) = \int_0^{\min(s,t)} \mathbf{e}^{(s-\tau)\mathbf{A}}\mathbf{B}\mathbf{B}^\top \mathbf{e}^{(t-\tau)\mathbf{A}^\top}\,\mathrm{d}\tau$$



Numerical example: submarine in a cavern

RKHS regression

$$\min_{\mathbf{x}(\cdot) \in \mathcal{S}_u} \quad \|\mathbf{x}(\cdot)\|_{K_1}^2$$

$$\text{s.t.}$$

$$z_1(t) \in [z_{\text{low},m}, z_{\text{up},m}],$$

$$\forall\, t \in [t_m - \delta_m, t_m + \delta_m], \forall m \in [M]$$

$$\mathcal{S}_u := \{\mathbf{x}(\cdot) \,|\, \mathbf{x}(\cdot) \in \mathcal{S} \text{ and } \mathbf{x}(0) = 0\}$$

$$\|\mathbf{x}(\cdot)\|_{K_1}^2 = \|\mathbf{u}(\cdot)\|_{L^2(0,1)}^2.$$

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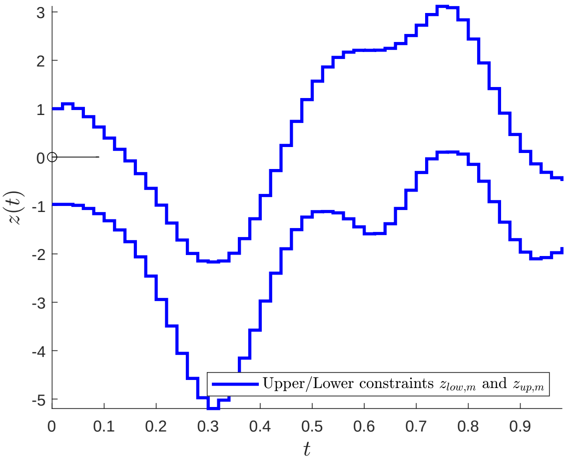


Numerical example: submarine in a cavern

RKHS regression

$$\min_{\mathbf{x}(\cdot)\in\mathcal{S}_u} \quad \|\mathbf{x}(\cdot)\|_{K_1}^2$$

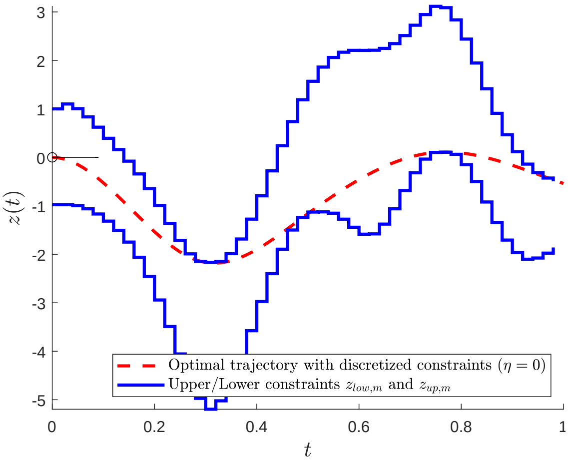
s.t.

$$z_1(t_m) \in [z_{\text{low},m}, z_{\text{up},m}],$$

$$\sout{\forall t \in [t_m - \delta_m, t_m + \delta_m]}, \forall m \in [M]$$

$$\bar{\mathbf{x}}(\cdot) = \sum_{m=1}^{M} K_1(\cdot, t_m)\mathbf{p}_m = \sum_{m=1}^{M} \alpha_m K_1(\cdot, t_m) e_m$$

$$K_1(s,t) = \int_0^{\min(s,t)} \mathbf{e}^{(s-\tau)\mathbf{A}}\mathbf{B}\mathbf{B}^\top \mathbf{e}^{(t-\tau)\mathbf{A}^\top} \mathrm{d}\tau$$



Numerical example: submarine in a cavern

RKHS regression

$$\min_{\mathbf{x}(\cdot)\in\mathcal{S}_u} \quad \|\mathbf{x}(\cdot)\|_{K_1}^2$$

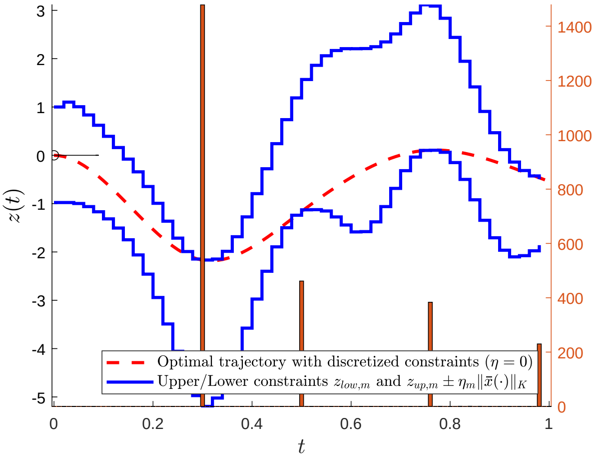
$$\text{s.t.}$$

$$z_1(t_m) \in [z_{\text{low},m}, z_{\text{up},m}],$$

$$\cancel{\forall t \in [t_m - \delta_m, t_m + \delta_m]},\ \forall m \in [M]$$

$$\bar{\mathbf{x}}(\cdot) = \sum_{m=1}^{M} K_1(\cdot, t_m)\mathbf{p}_m = \sum_{m=1}^{M} \alpha_m K_1(\cdot, t_m) e_m$$

$$K_1(s,t) = \int_0^{\min(s,t)} \mathbf{e}^{(s-\tau)\mathbf{A}}\mathbf{B}\mathbf{B}^\top \mathbf{e}^{(t-\tau)\mathbf{A}^\top} \mathrm{d}\tau$$



Numerical example: submarine in a cavern

RKHS regression

$$\min_{\mathbf{x}(\cdot) \in \mathcal{S}_u} \quad \|\mathbf{x}(\cdot)\|_{K_1}^2$$

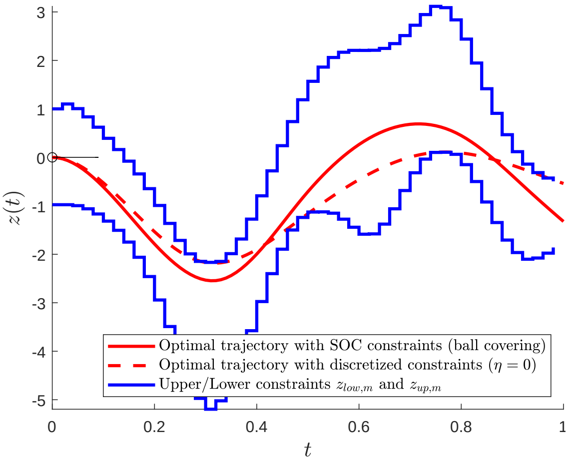
s.t.

$$z_1(t_m) \in [z_{\text{low},m}, z_{\text{up},m}] \pm \eta_m \|\mathbf{x}(\cdot)\|_{K_1},$$

$$\sout{\forall t \in [t_m - \delta_m, t_m + \delta_m]}, \forall m \in [M]$$

$$\bar{\mathbf{x}}(\cdot) = \sum_{m=1}^{M} K_1(\cdot, t_m)\mathbf{p}_m = \sum_{m=1}^{M} \alpha_m K_1(\cdot, t_m) e_m$$

$$K_1(s,t) = \int_0^{\min(s,t)} \mathbf{e}^{(s-\tau)\mathbf{A}} \mathbf{B}\mathbf{B}^\top \mathbf{e}^{(t-\tau)\mathbf{A}^\top} \mathrm{d}\tau$$



Van Loan's trick for time-invariant Gramians

Use matrix exponentials as in [Van Loan, 1978]

$$\exp\left(\begin{pmatrix} \mathbf{A} & \mathbf{Q}_c \\ 0 & -\mathbf{A}^\top \end{pmatrix} \Delta\right) = \begin{pmatrix} \mathbf{F}_2(\Delta) & \mathbf{G}_2(\Delta) \\ 0 & \mathbf{F}_3(\Delta) \end{pmatrix}$$

$$\hat{\mathbf{F}}_2(t) = e^{\mathbf{A}t}$$

$$\hat{\mathbf{F}}_3(t) = e^{-\mathbf{A}^\top t}$$

$$\hat{\mathbf{G}}_2(t) = \int_0^t e^{(t-\tau)\mathbf{A}} \mathbf{Q}_c e^{-\tau \mathbf{A}^\top} \mathrm{d}\tau$$

$$K_1(s,t) = \int_0^{\min(s,t)} e^{(s-\tau)\mathbf{A}} \mathbf{B}\mathbf{B}^\top e^{(t-\tau)\mathbf{A}^\top} \mathrm{d}\tau$$

Set $\mathbf{Q}_C = \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top$.

For $s \leq t,\ K_1(s,t) = \hat{\mathbf{G}}_2(s)\hat{\mathbf{F}}_2(t)^\top$

For $t \leq s,\ K_1(s,t) = \hat{\mathbf{F}}_2(s)\hat{\mathbf{G}}_2(t)^\top$

Dealing with an infinite number of constraints

No representer theorem for: $c(t)^\top x(t) \leq d, \forall t \in [0, T]$

Discretize on $\{t_m\}_{m \in [M]} \subset [0, T]$?

$$c(t_m)^\top x(t_m) \leq d, \forall m \in [\![1, M]\!]$$

No guarantees!

Dealing with an infinite number of constraints

No representer theorem for: $c(t)^\top x(t) \leq d, \forall t \in [0, T]$

Discretize on $\{t_m\}_{m\in[M]} \subset [0, T]$?

$$\eta_m \|x(\cdot)\|_K + c(t_m)^\top x(t_m) \leq d, \forall m \in [\![1, M]\!]$$

Second-Order Cone (SOC) constraints: $\{f \mid \|Af + b\|_K \leq c^\top f + d\}$

SOC comes from adding a buffer, $\eta_m > 0$, to a discretization, $\{t_m\}_{m\in[M]}$.

LP$\subset$ QP $\subset$ SOCP $\subset$ SDP

Dealing with an infinite number of constraints

No representer theorem for: $c(t)^\top x(t) \leq d, \forall t \in [0, T]$

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$$\eta_m \|x(\cdot)\|_K + c(t_m)^\top x(t_m) \leq d, \forall m \in [\![1, M]\!]$$

Second-Order Cone (SOC) constraints: $\{f \mid \|Af + b\|_K \leq c^\top f + d\}$

SOC comes from adding a buffer, $\eta_m > 0$, to a discretization, $\{t_m\}_{m\in[M]}$.

How to choose η_m? The choice $\eta_m \|x(\cdot)\|_K$ is related to continuity moduli:

Deriving SOC constraints through continuity moduli

Take $\delta \geq 0$ and t s.t. $|t - t_m| \leq \delta$

$$|c(t)^\top x(t) - c(t_m)^\top x(t_m)| = |\langle x(\cdot), K(\cdot, t)c(t) - K(\cdot, t_m)c(t_m)\rangle_K|$$
$$\leq \|x(\cdot)\|_K \underbrace{\sup_{\{t \,|\, |t-t_m|\leq\delta\}} \|K(\cdot, t)c(t) - K(\cdot, t_m)c(t_m)\|_K}_{\eta_m(\delta)}$$

$$\omega_m(x, \delta) := \sup_{\{t \,|\, |t-t_m|\leq\delta\}} |c(t)^\top x(t) - c(t_m)^\top x(t_m)| \leq \eta_m(\delta)\|x(\cdot)\|_K$$

For a covering $[0, T] = \bigcup_{m\in[M]}[t_m - \delta_m, t_m + \delta_m]$

$$“c(t)^\top x(t) \leq d, \forall t \in [0, T]“ \Leftarrow “c(t_m)^\top x(t_m) + \omega_m(x, \delta) \leq d, \forall m \in [M]“$$

Deriving SOC constraints through continuity moduli

Take $\delta \geq 0$ and t s.t. $|t - t_m| \leq \delta$

$$|c(t)^\top x(t) - c(t_m)^\top x(t_m)| = |\langle x(\cdot), K(\cdot, t)c(t) - K(\cdot, t_m)c(t_m)\rangle_K|$$
$$\leq \|x(\cdot)\|_K \underbrace{\sup_{\{t \,|\, |t-t_m|\leq\delta\}} \|K(\cdot, t)c(t) - K(\cdot, t_m)c(t_m)\|_K}_{\eta_m(\delta)}$$

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For a covering $[0, T] = \bigcup_{m\in[M]}[t_m - \delta_m, t_m + \delta_m]$

$$\text{“}c(t)^\top x(t) \leq d \quad, \forall t \in [0, T]\text{“} \Leftarrow \text{“}c(t_m)^\top x(t_m) + \eta_m\|x(\cdot)\| \leq d \quad, \forall m \in [M]\text{“}$$

$$\|K(\cdot, t)c(t) - K(\cdot, t_m)c(t_m)\|_K^2 := c(t)^\top K(t, t)c(t) + c(t_m)^\top K(t_m, t_m)c(t_m) - 2c(t_m)^\top K(t_m, t)c(t)$$

Since the kernel is smooth, for $c(\cdot) \in \mathcal{C}^0$, $\delta \to 0$ gives $\eta_m(\delta) \to 0$.

Deriving SOC constraints through continuity moduli

Take $\delta \geq 0$ and t s.t. $|t - t_m| \leq \delta$

$$|c(t)^\top x(t) - c(t_m)^\top x(t_m)| = |\langle x(\cdot), K(\cdot, t)c(t) - K(\cdot, t_m)c(t_m)\rangle_K|$$
$$\leq \|x(\cdot)\|_K \underbrace{\sup_{\{t \,|\, |t-t_m| \leq \delta\}} \|K(\cdot, t)c(t) - K(\cdot, t_m)c(t_m)\|_K}_{\eta_m(\delta)}$$

$$\omega_m(x, \delta) := \sup_{\{t \,|\, |t-t_m| \leq \delta\}} |c(t)^\top x(t) - c(t_m)^\top x(t_m)| \leq \eta_m(\delta)\|x(\cdot)\|_K$$

For a covering $[0, T] \subset \bigcup_{m \in [M]} [t_m - \delta_m, t_m + \delta_m]$

$$\text{“}c(t)^\top x(t) \leq d(t), \forall t \in [0, T]\text{“} \Leftarrow \text{“}c(t_m)^\top x(t_m) + \eta_m \|x(\cdot)\| \leq d_m, \forall m \in [M]\text{“}$$

with $d_m := \inf_{t \in [t_m - \delta_m, t_m + \delta_m]} d(t)$.

From affine state constraints to SOC constraints

Take (t_m, δ_m) such that $[0, T] \subset \cup_{m \in [\![1, N_P]\!]} [t_m - \delta_m, t_m + \delta_m]$, define

$$\eta_i(\delta_m, t_m) := \sup_{t \in [t_m - \delta_m, t_m + \delta_m] \cap [0, T]} \|K(\cdot, t_m)\mathbf{c}_i(t_m) - K(\cdot, t)\mathbf{c}_i(t)\|_K,$$

$$d_i(\delta_m, t_m) := \inf_{t \in [t_m - \delta_m, t_m + \delta_m] \cap [0, T]} d_i(t).$$

We have strengthened SOC constraints that enable a representer theorem

$$\eta_i(\delta_m, t_m)\|\mathbf{x}(\cdot)\|_K + \mathbf{c}_i(t_m)^\top \mathbf{x}(t_m) \leq d_i(\delta_m, t_m), \forall\, m \in [\![1, N_P]\!], \forall\, i \in [\![1, P]\!]$$

$$\Downarrow$$

$$\mathbf{c}_i(t)^\top \mathbf{x}(t) \leq d_i(t), \forall\, t \in [0, T], \forall\, i \in [\![1, P]\!]$$

Lemma (Uniform continuity of tightened constraints)

As $K(\cdot,\cdot)$ is UC, if $\mathbf{c}_i(\cdot)$ and $\mathbf{d}_i(\cdot)$ are $\mathcal{C}^0$-continuous, when $\delta \to 0^+$, $\eta_i(\cdot, t)$ converges to 0 and $d_i(\cdot, t)$ converges to $d_i(t)$, uniformly w.r.t. t.

SOC tightening of state-constrained LQ optimal control

Problem of time-varying linear quadratic optimal control with finite horizon and affine inequality state constraints

$$
\begin{aligned}
\min_{\mathbf{x}(\cdot) \in \mathcal{S}_{[t_0, T]}} \quad & \chi_{\mathbf{x}_0}(\mathbf{x}(t_0)) + g(\mathbf{x}(T)) + \|\mathbf{x}(\cdot)\|_K^2 \\
\text{s.t.} \quad & \\
& \mathbf{c}_i(t)^\top \mathbf{x}(t) \leq d_i(t), \; \forall\, t \in [t_0, T], \forall\, i \in [\mathcal{I}],
\end{aligned}
$$

SOC tightening of state-constrained LQ optimal control

Problem of time-varying linear quadratic optimal control with finite horizon and affine inequality state constraints with SOC tightening

$$\min_{\mathbf{x}(\cdot) \in \mathcal{S}_{[t_0,T]}} \quad \chi_{\mathbf{x}_0}(\mathbf{x}(t_0)) + g(\mathbf{x}(T)) + \|\mathbf{x}(\cdot)\|_K^2$$

$$\text{s.t.}$$

$$\eta_i(\delta_m, t_m)\|\mathbf{x}(\cdot)\|_K + \mathbf{c}_i(t_{i,m})^\top \mathbf{x}(t_{i,m}) \leq d_{i,m}, \ \forall\, m \in [M_i], \forall\, i \in [\mathcal{I}],$$

with $[t_0, T] \subset \bigcup_{m\in[M]}[t_m - \delta_m, t_m + \delta_m]$, and two values defined at each t_m

$$\eta_i(\delta_m, t_m) := \sup_{t \,\in\, [t_m - \delta_m, t_m + \delta_m]\cap[0,T]} \|K(\cdot, t_m)\mathbf{c}_i(t_m) - K(\cdot, t)\mathbf{c}_i(t)\|_K,$$

$$d_{i,m} := \inf_{t \,\in\, [t_m - \delta_m, t_m + \delta_m]\cap[t_0,T]} d_i(t).$$

Actually also works for ball constraints $\|\mathbf{x}(t)\|_p \leq 1$ and variations!

Main theoretical result in P.-C. Aubin, SICON, 2021

(H-gen) $\mathbf{A}(\cdot), \mathbf{Q}(\cdot) \in L^1$ and $\mathbf{B}(\cdot), \mathbf{R}(\cdot) \in L^2$, $\mathbf{c}_i(\cdot)$ and $d_i(\cdot) \in \mathcal{C}^0$.

(H-sol) $\mathbf{c}_i(t_0)^\top \mathbf{x}_0 < d_i(t_0)$ and there exists a trajectory $\mathbf{x}^\epsilon(\cdot) \in \mathcal{S}$ satisfying strictly the affine constraints, as well as the initial condition.[1]

(H-obj) $g(\cdot)$ is convex and continuous.

Theorem (∃/Approximation by SOC constraints, P.-C. Aubin, 2021)

Both the original problem and its strengthening have unique optimal solutions. For any $\rho > 0$*, there exists* $\bar{\delta} > 0$ *such that for all* $(\delta_m)_{m \in [\![1, N_0]\!]}$*, with* $[t_0, T] \subset \cup_{m \in [\![1, N_0]\!]} [t_m - \delta_m, t_m + \delta_m]$ *satisfying* $\bar{\delta} \geq \max_{m \in [\![1, N_0]\!]} \delta_m$*,*

$$\frac{1}{\gamma_K} \sup_{t \in [t_0, T]} \|\bar{\mathbf{x}}_\eta(t) - \bar{\mathbf{x}}(t)\| \leq \|\bar{\mathbf{x}}_\eta(\cdot) - \bar{\mathbf{x}}(\cdot)\|_K \leq \rho$$

with $\gamma_K := \sup_{t \in [0, T], \, \mathbf{p} \in \mathbb{B}_N} \sqrt{\mathbf{p}^\top K(t, t) \mathbf{p}}$.

[1] (H-sol) is implied for instance by an inward-pointing condition at the boundary.

Main practical result in P.-C. Aubin, SICON, 2021

Problem of time-varying linear quadratic optimal control with finite horizon and affine inequality state constraints with SOC tightening

$$\min_{\mathbf{x}(\cdot)\in\mathcal{S}_{[t_0,T]}} \quad \chi_{\mathbf{x}_0}(\mathbf{x}(t_0)) + g(\mathbf{x}(T)) + \|\mathbf{x}(\cdot)\|_K^2$$

$$\text{s.t.}$$

$$\eta_i(\delta_m, t_m)\|\mathbf{x}(\cdot)\|_K + \mathbf{c}_i(t_{i,m})^\top \mathbf{x}(t_{i,m}) \le d_{i,m},\ \forall\, m \in [M_i], \forall\, i \in [\mathcal{I}].$$

By the representer theorem, the optimal solution has the form

$$\bar{\mathbf{x}}(\cdot) = \sum_{j=0}^{P}\sum_{m=1}^{N_j} K(\cdot, t_{j,m})\bar{\mathbf{p}}_{j,m},$$

where $t_{0,1} = t_0$ and $t_{0,2} = T$, and the coefficients $(\bar{\mathbf{p}}_{j,m})_{j,m}$ solve a finite dimensional second-order cone problem.

Main practical result in P.-C. Aubin, SICON, 2021

More precisely, setting $t_{0,1} = t_0$ and $t_{0,2} = T$, the coefficients of the optimal solution $\bar{\mathbf{x}}(\cdot) = \sum_{j=0}^{P} \sum_{m=1}^{N_j} K(\cdot, t_{j,m})\bar{\mathbf{p}}_{j,m}$ solve

$$\min_{\substack{\gamma\in\mathbb{R}_+,\\ \mathbf{p}_{j,m}\in\mathbb{R}^N,\\ \alpha_{j,m}\in\mathbb{R}}} \quad \chi_{\mathbf{x}_0}\left(\sum_{j=0}^{P}\sum_{m=1}^{N_j} K(t_0, t_{j,m})\bar{\mathbf{p}}_{j,m}\right) + g\left(\sum_{j=0}^{P}\sum_{m=1}^{N_j} K(T, t_{j,m})\bar{\mathbf{p}}_{j,m}\right) + \gamma^2$$

$$\text{s.t.} \quad \gamma^2 = \sum_{i=0}^{P}\sum_{n=1}^{N_i}\sum_{j=0}^{P}\sum_{m=1}^{N_j} \mathbf{p}_{i,n}^\top K(t_{i,n}, t_{j,m})\mathbf{p}_{j,m},$$

$$\mathbf{p}_{j,m} = \alpha_{j,m}\mathbf{c}_j(t_m), \quad \forall\, m \in [\![1, N_j]\!], \forall\, j \in [\![1, P]\!],$$

$$\begin{matrix} \eta_i(\delta_{i,m}, t_{i,m})\gamma + \sum_{j=0}^{P}\sum_{m=1}^{N_j} \mathbf{c}_i(t_{i,m})^\top K(t_{i,m}, t_{j,m})\mathbf{p}_{j,m} & \forall\, m \in [\![1, N_i]\!], \\ \leq d_i(\delta_{i,m}, t_{i,m}), & \forall\, i \in [\![1, P]\!], \end{matrix}.$$

which can be written equivalently as a finite dimensional second-order cone problem (SOCP).

Take-home messages

“State-constrained LQ Optimal Control is a shape-constrained kernel regression.“

“Controlled trajectories have the adequate structure to use kernel methods, most of all for path-planning.“

“In general, positive definite kernels are much too linear to tackle nonlinear control problems $\rightarrow$ **Linearize**! “

Future work: Pushing RKHSs beyond/Revisiting LQR

For RKHSs

- Control constraints do not correspond to continuous evaluations
  $\hookrightarrow$ limits of RKHS pointwise theory (e.g. $x' = u \in L^2([0,T],[-1,1])$ a.e.)
- Successive linearizations of nonlinear system lead to changing kernels
  $\hookrightarrow$ a single kernel may not be sufficient (e.g. $x' = f_{[x_n(\cdot)]}x + f_{[u_n(\cdot)]}u$ a.e.)
- Non-quadratic costs for linear systems do not lead to Hilbert spaces
  $\hookrightarrow$ one may need Banach kernels (e.g. $\|\mathbf{u}(\cdot)\|^2_{L^2(0,T)} \to \|\mathbf{u}(\cdot)\|_{L^1(0,T)}$)

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- Non-quadratic costs for linear systems do not lead to Hilbert spaces
  $\hookrightarrow$ one may need Banach kernels (e.g. $\|\mathbf{u}(\cdot)\|^2_{L^2(0,T)} \to \|\mathbf{u}(\cdot)\|_{L^1(0,T)}$)

For control theory

- To each evaluation at time t corresponds a covector $p_t \in \mathbb{R}^Q$
  $\hookrightarrow$ Representer theorem well adapted for state constraints, but unsuitable for control constraints. Reverts the difficulty w.r.t. PMP approach.
- The Gramian of controllability generates trajectories
  $\hookrightarrow$ This allows for close-form solutions in continuous-time.

Future work and open questions

- Extending results to linear PDE control->done
- Extending results to Gramian of observability & Kalman filter->almost done

This talk summarizes

- *Linearly-constrained Linear Quadratic Regulator from the viewpoint of kernel methods*, Aubin, **SIAM J. on Control and Optimization, 2021**
- *Interpreting the dual Riccati equation through the LQ reproducing kernel*, Aubin, **Comptes Rendus. Mathématique, 2021**

The code is available at `https://github.com/PCAubin`

More to be found at `https://pcaubin.github.io/`

Thank you for your attention!

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IEEE Transactions on Automatic Control, 23(3):395–404.

Annex: Green kernels and RKHSs

Let D be a differential operator, D^* its formal adjoint. Define the Green function $G_{D^*D,x}(y) : \Omega \to \mathbb{R}$ s.t. $D^*D\, G_{D^*D,x}(y) = \delta_z(y)$ then, if the integrals over the boundaries in Green's formula are null, for any $f \in \mathcal{F}_k$

$$f(x) = \int_\Omega f(y) D^* D G_{D^*D,x}(y) dy = \int_\Omega Df(y) D G_{D^*D,x}(y) =: \langle f, G_{D^*D,x} \rangle_{\mathcal{F}_k},$$

so $k(x, y) = G_{D^*D,x}(y)$ [Saitoh and Sawano, 2016, p61]. For vector-valued contexts, e.g. $\mathcal{F}_K = W^{s,2}(\mathbb{R}^d, \mathbb{R}^d)$ and $D^*D = (1 - \sigma^2\Delta)^s$ component-wise, see [Micheli and Glaunès, 2014, p9].

Alternatively, in 1D, $D\, G_{D,x}(y) = \delta_z(y)$, the kernel associated to the inner product $\int_\Omega Df(y)Dg(y)dy$ for the space of f “null at the border“ writes as

$$k(x, y) = \int_\Omega G_{D,x}(z) G_{D,y}(z) dz$$

see [Berlinet and Thomas-Agnan, 2004, p286] and [Heckman, 2012].

Annex: IPC gives strictly feasible trajectories

(H-sol) $\mathbf{C}(0)\mathbf{x}_0 < \mathbf{d}(0)$ and there exists a trajectory $\mathbf{x}^\epsilon(\cdot) \in \mathcal{S}$ satisfying strictly the affine constraints, as well as the initial condition.

(H1) $\mathbf{A}(\cdot), \mathbf{B}(\cdot) \in \mathcal{C}^0$, $\mathbf{c}_i(\cdot), d_i(\cdot) \in \mathcal{C}^1$ and $\mathbf{C}(0)\mathbf{x}_0 < \mathbf{d}(0)$.

(H2) There exists $M_u > 0$ s.t. , for all $t \in [t_0, T]$ and $\mathbf{x} \in \mathbb{R}^Q$ satisfying $\mathbf{C}(t)\mathbf{x} \leq \mathbf{d}(t)$, and $\|\mathbf{x}\| \leq (1 + \|\mathbf{x}_0\|)e^{T\|\mathbf{A}(\cdot)\|_{L^\infty(t_0,T)} + TM_u\|\mathbf{B}(\cdot)\|_{L^\infty(t_0,T)}}$, there exists $\mathbf{u}_{t,x} \in M_u\mathbb{B}_M$ such that

$$\forall\, i \in \{j \mid \mathbf{c}_j(t)^\top \mathbf{x} = d_j(t)\},\ \mathbf{c}_i'(t)^\top \mathbf{x} - d_i'(t) + \mathbf{c}_i(t)^\top (\mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u}_{t,x}) < 0.$$

This is an inward-pointing condition (IPC) at the boundary.

Lemma (Existence of interior trajectories)

*If (H1) and (H2) hold, then **(H-sol)** holds.*

Annex: control proof main idea, nested property

$$\eta_i(\delta,t) := \sup \|K(\cdot,t)\mathbf{c}_i(t) - K(\cdot,s)\mathbf{c}_i(s)\|_K, \quad \omega_i(\delta,t) := \sup |d_i(t) - d_i(s)|,$$
$$d_i(\delta_m,t_m) := \inf d_i(s), \quad \text{over } s \in [t_m-\delta_m, t_m+\delta_m] \cap [t_0,T]$$

For $\overrightarrow{\boldsymbol{\epsilon}} \in \mathbb{R}_+^P$, the constraints we shall consider are defined as follows

$$\begin{aligned}
\mathcal{V}_0 &:= \{\mathbf{x}(\cdot) \in \mathcal{S} \,|\, \mathbf{C}(t)\mathbf{x}(t) \leq \mathbf{d}(t), \forall\, t \in [t_0,T]\},\\
\mathcal{V}_{\delta,\text{fin}} &:= \{\mathbf{x}(\cdot) \in \mathcal{S} \,|\, \overrightarrow{\boldsymbol{\eta}}(\delta_m,t_m)\|\mathbf{x}(\cdot)\|_K + \mathbf{C}(t_m)\mathbf{x}(t_m) \leq \mathbf{d}(\delta_m,t_m), \forall\, m \in [\![1,M_0]\!]\},\\
\mathcal{V}_{\delta,\text{inf}} &:= \{\mathbf{x}(\cdot) \in \mathcal{S} \,|\, \overrightarrow{\boldsymbol{\eta}}(\delta,t)\|\mathbf{x}(\cdot)\|_K + \overrightarrow{\boldsymbol{\omega}}(\delta,t) + \mathbf{C}(t)\mathbf{x}(t) \leq \mathbf{d}(t), \forall\, t \in [t_0,T]\},\\
\mathcal{V}_{\overrightarrow{\boldsymbol{\epsilon}}} &:= \{\mathbf{x}(\cdot) \in \mathcal{S} \,|\, \overrightarrow{\boldsymbol{\epsilon}} + \mathbf{C}(t)\mathbf{x}(t) \leq \mathbf{d}(t), \forall\, t \in [t_0,T]\}.
\end{aligned}$$

Proposition (Nested sequence)

Let $\delta_{max} := \max_{m\in[\![1,M_0]\!]} \delta_m$. For any $\delta \geq \delta_{max}$, if, for a given $y_0 \geq 0$, $\epsilon_i \geq \sup_{t\in[t_0,T]}[\eta_i(\delta,t)y_0 + \omega_i(\delta,t)]$, then we have a nested sequence

$$(\mathcal{V}_{\overrightarrow{\boldsymbol{\epsilon}}} \cap y_0\mathbb{B}_K) \subset \mathcal{V}_{\delta,inf} \subset \mathcal{V}_{\delta,fin} \subset \mathcal{V}_0.$$

Only the simpler $\mathcal{V}_{\overrightarrow{\boldsymbol{\epsilon}}}$ constraints matter!

Numerical example 2: constrained pendulum - definition

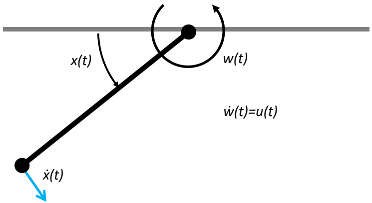
Constrained pendulum when controlling the third derivative of the angle

$$\min_{x(\cdot),w(\cdot),u(\cdot)} -\dot{x}(T) + \lambda \|u(\cdot)\|^2_{L^2(0,T)} \qquad \lambda \ll 1$$

$$x(0) = 0.5, \quad \dot{x}(0) = 0, \quad w(0) = 0, \quad x(T/3) = 0.5, \quad x(T) = 0$$

$$\ddot{x}(t) = -10\,x(t) + w(t), \quad \dot{w}(t) = u(t), \text{ a.e. in } [0, T]$$

$$\dot{x}(t) \in [-3, +\infty[, \quad w(t) \in [-10, 10], \forall\, t \in [0, T]$$



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Converting affine state constraints to SOC constraints, applying rep. thm

$$\begin{aligned} \eta_{\dot{x}}\|\mathbf{x}(\cdot)\|_K - \dot{x}(t_m) &\leq 3, \\ \eta_w\|\mathbf{x}(\cdot)\|_K + w(t_m) &\leq 10, \\ \eta_w\|\mathbf{x}(\cdot)\|_K - w(t_m) &\leq 10 \end{aligned}$$

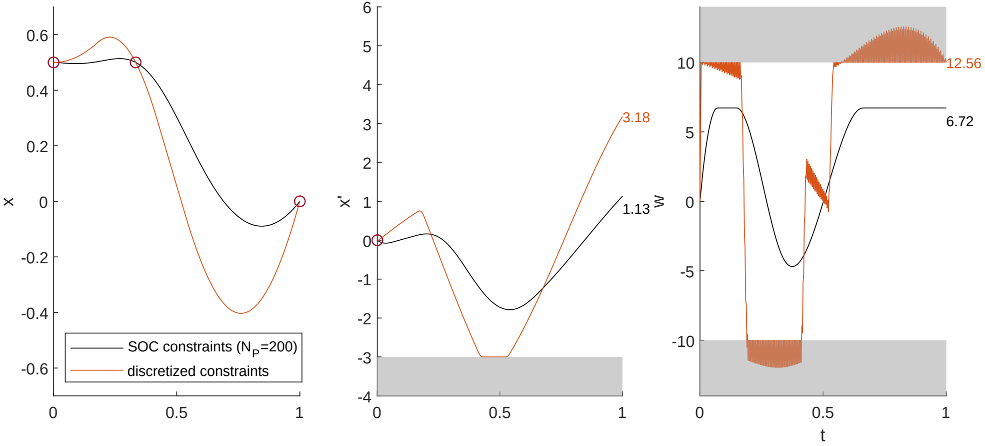
$$\begin{aligned} \bar{\mathbf{x}}(\cdot) = {} & K(\cdot, 0)\mathbf{p}_0 + K(\cdot, T/3)\mathbf{p}_{T/3} \\ & + K(\cdot, T)\mathbf{p}_T + \sum_{m=1}^{M} K(\cdot, t_m)\mathbf{p}_m \end{aligned}$$

Most of computational cost is related to the "controllability Gramians"
$K_1(s,t) = \int_0^{\min(s,t)} \mathbf{e}^{(s-\tau)\mathbf{A}}\mathbf{B}\mathbf{B}^\top\mathbf{e}^{(t-\tau)\mathbf{A}^\top}\mathrm{d}\tau$ which we have to approximate.

Numerical example 2: constrained pendulum - illustration

Optimal solutions of the constrained pendulum "path-planning" problem.
Red circles: equality constraints. Grayed areas: constraints over $[0, T]$.

Angle $x(\cdot)$ Velocity $\dot{x}(\cdot)$ Couple $w(\cdot)$



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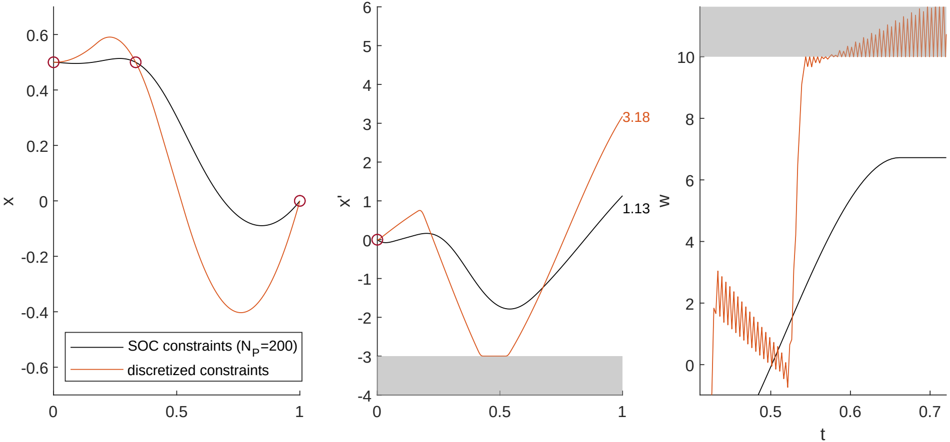

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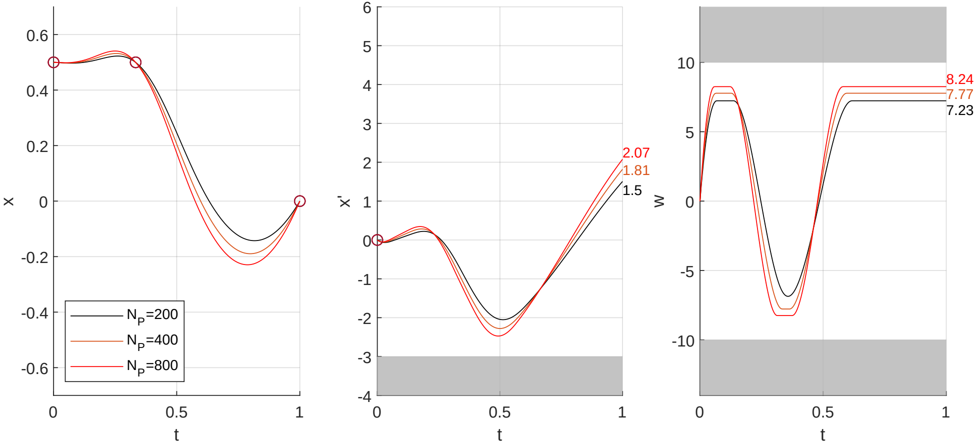

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