Lipschitz regularity of the minimum time function of differential inclusions with state constraints

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Abstract

For control systems, the local regularity of the minimum time function τ_{min} in the absence of state constraints has been extensively studied and related both to inward-pointing conditions and to small-time controllability in the neighborhood of a closed target C. In the presence of state constraints, assessing this regularity is crucial to ensure the existence of solutions when perturbing the initial condition. In this paper, we prove, without imposing the inclusion $C \subset \operatorname{Int} K$, that, for differential inclusions with closed state constraints K and under general assumptions, τ_{min} is locally Lipschitz continuous on its domain which is open in K. We discuss as well extensions to nonautonomous systems and to point targets.

Keywords: Time optimal control, Differential inclusions, State constraints, Lipschitz continuity

1 Introduction

Studying the regularity of the minimum time function finds its motivation in reachability problems. Let K and C be two closed subsets of $\mathbb{R}^n$ and consider a control system with initial condition $x_0 \in K$

$$\begin{cases} x'(t) = f(x(t), u(t)), & u(t) \in U, \\ x(0) = x_0, \end{cases} \tag{1}$$

where U is a compact subset of $\mathbb{R}^m$, the control $u(\cdot)$ is a measurable function and $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is sufficiently smooth. The state-constrained time optimal control problem consists in finding the minimum time $\tau_{min}(x_0)$ to reach C along solutions of (1) staying in K. Assessing the regularity of τ_{min} allows to answer several questions. For instance, let a time-optimal state-constrained solution $x_{\text{ref}}(\cdot)$ of system (1) be given. Take as initial condition a point x_1 in a neighborhood of the trajectory set $x_{\text{ref}}([0, \tau_{min}(x_0)])$. Under what conditions can x_1 be steered to the target set C while respecting the state constraints K?

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How long would it take compared to $\tau_{min}(x_0)$?

More generally, consider the autonomous differential inclusion[1] with initial condition $x_0 \in K$:

$$x'(t) \in F(x(t)) \qquad x(0) = x_0 \tag{2}$$

where $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a set-valued map taking closed, nonempty values. Below, an F-trajectory $x(\cdot)$ on a time interval $[0, T]$ designates an absolutely continuous function satisfying $x'(t) \in F(x(t))$ a.e. on $[0, T]$. A trajectory is called "feasible" if $x([0, T]) \subset K$.

The capture basin $\mathrm{Capt}_F(K, C)$ is the set of all points $x_0 \in K$ such that there exists $T \geq 0$ and a feasible F-trajectory starting from x_0 and reaching the target set C at time T. For a given $x_0 \in K$, we denote the infimum of such T by $\tau_{min}(x_0)$. By convention, $\tau_{min}(x_0) = +\infty$ if $x_0 \notin \mathrm{Capt}_F(K, C)$. By analogy with the control system (1), this defines the minimum time extended function $\tau_{min} : K \to [0, +\infty]$ associated with the target set C, dynamics F and state-constraints K. The Lipschitzianity of the extended function $\tau_{min}(\cdot)$ thus depends on the "Lipschitz regularity" of both F, K and C.

In order to exhibit Lipschitz dependence of solutions on initial conditions (through the renowned Filippov's theorem), it is classical to suppose the (local) Lipschitzianity of F. On the other hand, the local Lipschitz continuity of $\tau_{min}(\cdot)$ on its domain $\mathrm{Capt}_F(K, C)$, in the case without state constraints, has been related to strict inward-pointing conditions on the boundary ∂C of the target since the 70s (see e.g. [4, Chapter 8] for a modern presentation and the bibliography therein). More recently, it has been shown in [3] that strict inward-pointing conditions on the boundary ∂K ensure L^∞-distance estimates between arbitrary F-trajectories and the set of feasible ones. These three ingredients allow us to prove in this paper the openness of $\mathrm{Capt}_F(K, C)$ in K and the local Lipschitz continuity of $\tau_{min}(\cdot)$.

Inquiries on the regularity of the minimum time function using constraint qualifications (albeit without state-constraints) go back at least to the early 70s. As a matter of fact, the latter regularity can be related to a controllability property in the vicinity of the target. For control systems, (local) Lipschitz continuity was already obtained for C^2-regular target sets in a general setting of differential games in [6, Theorem 5] and in a neighborhood of point targets in [8, Theorem 4.1]. Through viscosity solutions theory, Lipschitz continuity on $\mathrm{Capt}_F(\mathbb{R}^n, C)$ was then shown for compact piecewise-C^2 targets in [2, Theorem 5.4] and in a neighborhood of general closed target sets in [10, Corollary 3.7]. Based on nonsmooth analysis and moving to differential inclusions, [11, Theorem 3.1] showed the Lipschitz continuity of τ_{min} on $\mathrm{Capt}_F(\mathbb{R}^n, C)$ for nonautonomous convex-valued F (measurable in time), while [12, Theorem 6.1] revisited the regularity in a neighborhood of a closed target set C. Finally, for state-constrained nonautonomous control systems (Lipschitz in time), the local Lipschitz continuity of τ_{min} was shown in [5, Theorem 3.8] under assumptions similar to ours, although more stringent as they bore on $f(x, U)$ rather than its convex

[1] We shall consider nonautonomous systems in Section 3.1, when F is also locally Lipschitz in the time variable.

hull $\mathrm{co}\,(f(x,U))$. We also discard the strong assumption of [5] of having C interior to K. Though we only consider compact-valued F, we have to mention that the minimum time problem has also been studied for control-affine systems where $U = \mathbb{R}^m$ as in [7] and references therein.

In this article on state-constrained differential inclusions, under general assumptions on F, K and C, and a convexified version of the inward-pointing conditions, we prove the local Lipschitz continuity of τ_{min}, that had been shown for control systems without state constraints. Such a property is a first step in studying nonlinear controllability with nonsmooth state constraints. Furthermore our results encompass those of [5] on nonautonomous control systems, as presented in Section 3.1. In Section 3.3, we show as well that, for point targets that are interior to the constraints, the classical small-time controllability condition is sufficient to retrieve the local Lipschitzianity of τ_{min} on its domain $\mathrm{Capt}_F(K,C)$.

2 Main results

Notation: We denote by $\mathbb{B}$ the closed unit ball in $\mathbb{R}^n$, by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$ and by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the Euclidean norm and scalar product. We denote by $\mathbb{R}_+$ the set of nonnegative real numbers. We write $\Pi_K(x)$ for the (possibly set-valued) projection of a point x into K. The function $d_K(\cdot)$ designates the distance to K. The set $\mathrm{Int}\,K$ stands for the interior of K and the set ∂K for its boundary. We denote by $T_K(x)$ (resp. $N_K(x)$) the Clarke tangent (resp. normal) cone to the subset K at point x. We use the notation $\mathrm{co}\,F$ for the set-valued map that maps x to the convex hull of $F(x)$.

Assumption 1. *General assumptions*

$$\begin{cases} F \text{ takes closed, nonempty values on } \mathbb{R}^n, \\ C \text{ and } K \text{ are two closed nonempty subsets of } \mathbb{R}^n. \end{cases}$$

Assumption 2. *Sublinear growth and local Lipschitz continuity of F*

$$\exists\, A \geq 0,\ \forall\, x \in \mathbb{R}^n,\ F(x) \subset A(1 + \|x\|)\mathbb{B}$$

$$\forall\, R > 0,\ \exists\, k_F \geq 0,\ \forall\, x, y \in R\mathbb{B},\ F(y) \subset F(x) + k_F\|x - y\|\mathbb{B}$$

Assumption 3. *Strict inward-pointing condition on ∂K*

$$\forall\, x \in \partial K,\ \mathrm{co}\,F(x) \cap \mathrm{Int}\,T_K(x) \neq \emptyset$$

Assumption 4. *Strict inward-pointing condition on $\partial C \cap K$*

$$\forall\, x \in \partial C \cap K,\ \mathrm{co}\,F(x) \cap \mathrm{Int}\,T_K(x) \cap \mathrm{Int}\,T_C(x) \neq \emptyset$$

Remark 1. It stems directly from [9, Theorem 2] that for any $x \in C \cap K$ such that $\mathrm{Int}\,T_K(x) \cap \mathrm{Int}\,T_C(x) \neq \emptyset$, we have

$$\mathrm{Int}\,T_{C\cap K}(x) = \mathrm{Int}\,T_K(x) \cap \mathrm{Int}\,T_C(x).$$

Remark 2. For any closed subset K of $\mathbb{R}^n$, at a given $x \in \partial K$, for $v \in \mathbb{R}^n$ and a fixed $\epsilon > 0$, we have:

$$(v + \epsilon\mathbb{B}) \subset T_K(x) \Leftrightarrow \max_{n \in N_K(x)} \left\langle v, \frac{n}{\|n\|} \right\rangle \leq -\epsilon \tag{3}$$

The existence at $x \in \partial K$ of such $v \in \operatorname{co} F(x)$ and ϵ is implied by Assumption 3. Relation (3) allows us to juggle the two translations of strict inward-pointing conditions (based either on the normal cone or on the Clarke tangent cone to the sets). As a matter of fact, while the tangent cone is suitable to build trajectories staying in a set, the normal cone is easier to use when designing trajectories outside a set. In our case, we have both to stay in K and to reach C, leading us to use both perspectives.

Remark 3. Assumptions 3 and 4 actually have deeper implications on the regularity of the sets K and C. If for every $x \in \partial K$, $\operatorname{Int} T_K(x) \neq \emptyset$ (i.e. K is *wedged*), then ∂K is epi-Lipschitzian (it can be represented locally as the epigraph of a Lipschitz function after a nonsingular linear transform, see [9, Theorem 3]), and the same observation applies to $\partial C \cap K$. Furthermore, the characterization of the interior of the Clarke tangent cone (e.g. [9, Theorem 2]) implies that both K and $C \cap K$ are the closure of their interiors. Therefore Assumptions 3 and 4 implicitly require ∂K and $\partial C \cap K$ to be "Lipschitzian surfaces" and K and $C \cap K$ to be the closure of open sets of $\mathbb{R}^n$. In particular C cannot be a point target. Notice that we did not require C to be a subset of K, unlike [5] where the inclusion $C \subset \operatorname{Int} K$ was assumed.

Remark 4. Owing to [1, Theorem 10.1.6], the above assumptions also imply that K is viable under $\operatorname{co} F$ (i.e. for any $x_0 \in K$ there exists a feasible $\operatorname{co} F$-trajectory defined on $[0, +\infty[$ starting at x_0). [3, Theorem 2.3] shows that K is even viable under F.

The two following theorems are the main results of this article. Examples 1, 2 and 3 illustrate when they apply.

Theorem 1. *Under Assumptions 1, 2 and 3, Assumption 4 implies the following property of the minimum time function τ_{min} to reach the target C subject to the state constraints K:*

$$\forall R > 0,\ \exists \delta > 0,\ k > 0,\ \forall x \in (C + \delta\mathbb{B}) \cap R\mathbb{B} \cap K,\ \tau_{min}(x) \leq kd(x) := kd_{C \cap K}(x) \tag{4}$$

where, by convention, $d_\emptyset(x) = +\infty$.

Theorem 2. *Under Assumptions 1, 2, 3 and 4, $Capt_F(K, C)$ is open in K and τ_{min} is locally Lipschitz continuous on $Capt_F(K, C)$.*

Example 1. Consider a two-dimensional simplified lunar landing module with orientable exhaust nozzle, subject to the lunar gravity pulling downwards. Define the constraints as being above the surface of the moon $K = \mathbb{R} \times \mathbb{R}_+$ and the target as a box $C = [-\epsilon, \epsilon] \times [0, \epsilon]$ with $\epsilon > 0$. When close to the ground, the pilot is gradually allowed to activate emergency boosters, strong enough to overcome the gravity, so that its dynamics are

$$F(x) := \underbrace{(0, -1)}_{\text{gravity}} + \underbrace{[-1, 1] \times \{0\}}_{\text{nozzle}} + \underbrace{\max(0, 1/2 - x_2)\{(0, 0), (0, 4)\}}_{\text{boosters}} \subset \mathbb{R}^2.$$

It can be easily checked that the triplet (F, K, C) satisfies all the Assumptions 1-4, and that $\operatorname{Capt}_F(K, C) = K$.

Example 2. Define a scalar potential $g_y(x) = \max(0, \min(1 - \|x - y\|, \|x - y\|))$ centered at y. Consider a navigation problem: a child in a two-dimensional stream $\mathbb{R} \times [-2, 2]$ wants to reach an aquatic slide $C = \mathbb{B}((0, 0), 1/2)$ which creates a local whirl attractor. This defines the following dynamics

$$x' \in F(x) := \underbrace{(1,0)}_{\text{flow}} + \underbrace{\mathbb{B}((0,0), \frac{1}{2})}_{\text{swimmer's controls}} \underbrace{- 2g_{(0,0)}(x)\frac{x}{\|x\|}}_{\text{whirl attractor}} \subset \mathbb{R}^2,$$

with the whirl continuously extended as $(0, 0)$ at $x = (0, 0)$. A wave generator centered at $x_K = (-2, 0)$ is added. The constraint set is defined as $K = (\mathbb{R} \times [-2, 2]) \setminus \mathrm{Int}(\mathbb{B}(x_K, 1/2))$ and the new dynamics, when the wave generator is on, are

$$x' \in \tilde{F}(x) := F(x) + \underbrace{g_{x_K}(x)\frac{x - x_K}{\|x - x_K\|}}_{\text{wave generator}}.$$

One can verify that $(\tilde{F}, K, C)$ satisfies the assumptions of Theorem 2. One can also check that $\mathrm{Capt}_{\tilde{F}}(K, C) \subsetneq K$ is a proper open subset of K. On the other hand, when the wave generator is off, (F, K, C) does not satisfy the strict-inward pointing condition at $(-5/2, 0) \in \partial K$.

3 Discussion on the main results

3.1 Nonautonomous systems

Consider the nonautonomous differential inclusion with initial condition $x_0 \in K$:

$$x'(t) \in F(t, x(t)) \qquad x(0) = x_0$$

where $F : \mathbb{R}_+ \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a set-valued map taking closed, nonempty values on $\mathbb{R}_+ \times \mathbb{R}^n$. Below, we shall make the same assumption as in [5] that $F(\cdot, x)$ is locally Lipschitz continuous to show that Theorem 2 encompasses the results of [5] for nonautonomous systems. This may appear as a restriction[2], however there are comparatively much less attempts at considering nonautonomous systems in their full generality (i.e. when $F(\cdot, x)$ is merely measurable). For differential inclusions and a moving target $C(t)$ but without state constraints, we should mention [11]. For differential inclusions with fixed state constraints K but without target, we refer to [3]. As a midway, [5] considered a nonautonomous control system with compact control set and fixed K and C.

Assumption 5. *Sublinear growth and local Lipschitz continuity of nonautonomous F*

$$\exists\, A \geq 0,\ \forall\, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n,\ F(t, x) \subset A(1 + \|x\|)\mathbb{B}$$

$$\forall\, R > 0, \forall\, T \geq 0\ \exists\, k_F \geq 0,\ \forall\, x, y \in R\mathbb{B},\ \forall\, t, s \in [0, T]\ F(s, y) \subset F(t, x) + k_F(\|x - y\| + |t - s|)\mathbb{B}$$

[2]Time-Lipschitzianity was acknowledged in [5] as too restrictive. Nonetheless [5] also considered the case of $F(t, x) = c(t, x)\mathbb{B}$ where c is a bounded scalar function, globally Lipschitz in x and merely measurable in t.

By augmenting the dynamics, our results encompass those of [5]. As a matter of fact, let K and C be two closed subsets of $\mathbb{R}^n$. We define the augmented system $\hat{x}'(t) \in \hat{F}(\hat{x})$ under state constraints $\hat{K}$ and with target $\hat{C}$ as

$$\begin{cases} \hat{K} = \mathbb{R}_+ \times K,\ \hat{C} = \mathbb{R}_+ \times C \\ \hat{x}(\cdot) = (\tau(\cdot), x(\cdot)) \\ \hat{F}(\hat{x}) = (1, F(\hat{x})) \\ \hat{x}(0) = (0, x_0) \end{cases}$$

As $\tau'(\cdot) = 1$ and $\hat{K}$ and $\hat{C}$ are unbounded on the right in the time variable, Assumptions 1, 3, 4 and 5 on F, K and C jointly imply Assumptions 1, 2, 3, 4 bearing on $\hat{F}$, $\hat{K}$ and $\hat{C}$. So we may apply Theorem 2 to the augmented system. This shows that $Capt_{\hat{F}}(\hat{K}, \hat{C})$ is open in $\hat{K}$ for the relative Euclidean topology of $\mathbb{R}_+ \times \mathbb{R}^n$ which is stronger than the Cartesian product topology. Furthermore the minimum time functions coincide for the two systems. We obtain therefore the conclusions of Theorem 2 for the original nonautonomous system.

In the proof of Theorems 1 and 2, a key ingredient, namely [3, Theorem 2.3], was proven for nonautonomous differential inclusions when $F(\cdot, x)$ is absolutely continuous from the left. As a consequence, our results could eventually be extended to this class of systems.

3.2 Weakening the hypotheses

Among the other hypotheses, few could be relaxed. Indeed Assumptions 1 and 2 are fairly general and related to the global existence of solutions of (2) and to their Lipschitz dependence on initial conditions. Hence, they could hardly be weakened when we seek Lipschitz regularity.

Assumption 4 cannot be replaced by Assumption 3 expressed for C instead of K, as shown in (counter)-Example 3, so we require a jointly inward-pointing condition on K and C.

Example 3. Consider a modification of Example 1, where the controller over the nozzle of the landing module broke. The dynamics are now $x' \in F(x) = \{(0, \pm 1)\} \subset \mathbb{R}^2$, the constraints are still $K = \mathbb{R} \times \mathbb{R}_+$, and the target is now $C = Hyp(x_1 \mapsto 1 - x_1^2)$, where Hyp stands for the hypograph. It is clear that K and C satisfy inward-pointing conditions with respect to F, while $C \cap K$ does not. Furthermore $\mathrm{Capt}_F(K, C) = K \cap ([-1, 1] \times \mathbb{R})$ is closed in K and τ_{min} is discontinuous on K. In other words, for some initial conditions, an infinitesimal perturbation may impede reaching the target.

We cannot either drop altogether assuming inward-pointing conditions on C. Indeed, the problem without state constraints is a special case of a state-constrained problem. So the necessary and sufficient conditions from [11, Theorem 2.1] have to remain valid. Let $\Gamma = \mathrm{Graph}\, C(\cdot)$ where $(C(t))_{t \geq 0}$ is a moving target. In the simpler case of $F(\cdot, x)$ being continuous and $F(t, \cdot)$ being locally Lipschitz, the necessary condition for the local Lipschitzianity of the minimum time, as shown in [11, Corollary 2.1, Theorem 3.1] can be stated as follows[3]: for every compact set $G \subset \mathbb{R}_+ \times \mathbb{R}^n$, there exists $\epsilon > 0$ such that, for

[3]The original condition of [11] bore on the proximal normal cone to Γ but, by taking the limit and the closed convex hull, it can be restated for the Clarke normal cone.

every $(t,x) \in \partial\Gamma \cap G$,

$$\sup_{(p^0,p)\in N_\Gamma(t,x)\cap\mathbb{S}^n} \left(\min_{v\in F(t,x)} \left(p^0 + \langle p, v\rangle \right) \right) \leq -\epsilon \tag{5}$$

Condition (5) would then have to be satisifed. On the other hand, if the set K is invariant under F (i.e. for any $x_0 \in K$ all the solutions of the differential inclusion (2) are feasible), then no inward-pointing condition on K is required as all the trajectories are already feasible. Nevertheless we still need a strict inward-pointing condition on C, such as (5).

3.3 Considering point targets

Originally, [8] considered point targets (i.e. $C = \{\bar{x}\}$) and devised a necessary and sufficient condition ($0 \in \mathrm{Int}(\mathrm{co}\, F(\bar{x}))$) for the Lipschitz continuity of the minimum time in a neighborhood of $\bar{x}$, in the case without constraints. Since then, research focused mainly on inward-pointing conditions which preclude point targets as discussed in Remark 3. We show below that for point targets $\bar{x} \in \mathrm{Int}\, K$, we still have the results of Theorems 1 and 2. Whenever $\bar{x} \in \partial K$, it is still an open question as to formulating sufficient conditions for local Lipschitzianity of the minimum time.

We begin with an extension of [8, Theorem 4.1] to differential inclusions without state constraints.

Proposition 1. *Let $\bar{x} \in \mathbb{R}^n$ and suppose that $0 \in \mathrm{Int}(\mathrm{co}\, F(\bar{x}))$. Then, under Assumptions 1 and 2, the minimum time function τ_{min} to reach the target $\bar{x}$ without state constraints has the following property:*

$$\exists\, \delta > 0,\ k > 0,\ \forall\, x \in (\bar{x} + \delta\mathbb{B}),\ \tau_{min}(x) \leq k\|x - \bar{x}\| \tag{6}$$

A proof of Proposition 1 appears in the next section. We first state our result for point targets $\bar{x} \in \mathrm{Int}\, K$.

Proposition 2. *Let $\bar{x} \in \mathrm{Int}\, K$ and suppose that $0 \in \mathrm{Int}(\mathrm{co}\, F(\bar{x}))$. Then, under Assumptions 1, 2 and 3, τ_{min} satisfies* (6) *and is locally Lipschitz continuous on $Capt_F(K, \{\bar{x}\})$, which is open in K.*

Proof of Proposition 2: Fix $\eta > 0$ such that $(\bar{x} + \eta\mathbb{B}) \subset \mathrm{Int}\, K$. Apply Proposition 1 to the system (2) without constraints, this gives $\delta_0 > 0$ and $k > 0$ such that, for any $\epsilon > 0$ and any $x \in (\bar{x} + \delta_0\mathbb{B})$, there exists $\tau \in [0, k\|x - \bar{x}\| + \epsilon]$ and an F-trajectory $y(\cdot)$ defined on $[0, \tau]$ satisfying:

$$y(0) = x \qquad y(\tau) = \bar{x} \tag{7}$$

Let $A > 0$ be as in Assumption 2. Fix any $\delta \in]0, \delta_0]$ and $\epsilon > 0$ satisfying:

$$(\delta + (k\delta + \epsilon)\cdot(A + A\|\bar{x}\|))\, e^{A(k\delta+\epsilon)} \leq \eta$$

Take $x \in (\bar{x} + \delta\mathbb{B})$ and consider τ and $y(\cdot)$ as in (7). Then for any $t \in [0, \tau]$ we have:

$$\begin{aligned}\|\bar{x} - y(t)\| &\leq \|\bar{x} - x\| + \|x - y(t)\| \leq \delta + \int_0^t \|y'(s)\|ds \\ &\leq \delta + \int_0^t (A + A\|\bar{x}\| + A\|\bar{x} - y(s)\|)ds \\ &\leq \delta + \tau(A + A\|\bar{x}\|) + A\int_0^t \|\bar{x} - y(s)\|ds\end{aligned}$$

Using Gronwall's lemma, we get that for any $t \in [0, \tau]$, as $\tau \leq k\delta + \epsilon$

$$\|\bar{x} - y(t)\| \leq (\delta + \tau(A + A\|\bar{x}\|))\, e^{A\tau} \leq \eta$$

Consequently $y([0, \tau]) \subset\subset \bar{x} + \eta\mathbb{B} \subset \operatorname{Int} K$, so $y(\cdot)$ is feasible. As ϵ is arbitrary, we get that the minimum time with state constraints coincides on $(\bar{x} + \delta\mathbb{B})$ with the minimum time without state constraints, and that both are therefore Lipschitz continuous on $(\bar{x} + \delta\mathbb{B})$. The results of Theorem 2 follow immediately, which concludes the proof.

■

4 Proofs

Proof of Theorem 1: This proof is partially inspired by [4, pp. 239-243] where the result was proven for a control system without state constraints. The proof differs however due to Assumption 4 which is weaker than in [4] as it bears on $\operatorname{co} F$ rather than F, and as we consider a general differential inclusion rather than a control system. Moreover the presence of constraints requires to design feasible trajectories (i.e. respecting the state constraints). This leads to applying both the celebrated relaxation theorem and a "correction" theorem [3, Theorem 2.3] to build F-trajectories staying in K.

Fix any $R > 0$. Let $k_F > 0$ such that:

$$\forall\, x, y \in 2R\mathbb{B},\ F(y) \subset F(x) + k_F\|x - y\|\mathbb{B}$$

Let $c_F := 5nk_F$ and define

$$M = n + \sup_{x \in 2R\mathbb{B}} \sup_{v \in F(x)} \|v\|.$$

By Assumptions 3 and 4 and [3, Lemma 5.3], there exists $\epsilon \in]0, 1]$, $\eta_0 > 0$ such that:

$$\begin{cases} \forall\, x \in (\partial(C \cap K) + \eta_0\mathbb{B}) \cap 2R\mathbb{B} \cap C \cap K,\ \exists\, v \in \operatorname{co} F(x), \\ \quad \forall y \in (x + \eta_0\mathbb{B}) \cap C \cap K,\ y + [0, \epsilon](v + \epsilon\mathbb{B}) \subset C \cap K \end{cases} \tag{8}$$

$$\begin{cases} \forall\, x \in (\partial C \cap \partial K + \eta_0\mathbb{B}) \cap 2R\mathbb{B} \cap C \cap K,\ \exists\, v \in \operatorname{co} F(x), \\ \quad \forall y \in (x + \eta_0\mathbb{B}) \cap K,\ y + [0, \epsilon](v + \epsilon\mathbb{B}) \subset K \\ \quad \forall z \in (x + \eta_0\mathbb{B}) \cap C \cap K,\ z + [0, \epsilon](v + \epsilon\mathbb{B}) \subset C \cap K. \end{cases} \tag{9}$$

In order to use Gronwall's lemma later on, we require a technical condition on ϵ, which has to be chosen small enough as to satisfy:

$$\epsilon e^{c_F\epsilon/(16M^2)} \leq 8MR \tag{10}$$

Since $\mathrm{co}(F)$ is locally Lipschitz continuous, there exists $\eta_1 > 0$, such that for every $\tilde{x} \in \partial C \cap K \cap 2R\mathbb{B}$ and $\tilde{v} \in \mathrm{co}\, F(\tilde{x})$

$$\forall\, x \in (\tilde{x} + \eta_1 \mathbb{B}),\ \exists\, v \in \mathrm{co}\, F(x),\ \|v - \tilde{v}\| \leq \epsilon/4. \tag{11}$$

Let $\eta := \min(\eta_0, \eta_1, R)/2$. For any $\delta \in]0, \eta/3]$, we now define the following sets and quantities:

$$\Delta = \partial C \cap \partial K \cap 2R\mathbb{B} \qquad \tilde{C} = (\partial C \cap K \cap 2R\mathbb{B}) \setminus (\Delta + \frac{\eta}{2}\mathbb{B}) \qquad \tilde{\rho} = \inf_{x \in \tilde{C}} d_{\partial K}(x)$$
$$C_\delta = \partial C \cap K + \delta\mathbb{B} \qquad \tilde{C}_\delta = \left(C_\delta \cap K \cap \frac{3R}{2}\mathbb{B}\right) \setminus (\Delta + \eta\mathbb{B}) \qquad \tilde{\rho}_\delta = \inf_{x \in \tilde{C}_\delta} d_{\partial K}(x)$$

where, by convention, $\emptyset + \mathbb{B} = \emptyset$. The initial conditions x_0 of interest are in $C_\delta \cap K \cap R\mathbb{B}$, hence they either belong to $(\Delta + \eta\mathbb{B})$ or to $\tilde{C}_\delta$.

Claim 1. Whenever $\tilde{C}_\delta \neq \emptyset$, $\tilde{\rho} > 0$ is finite, independently of Δ being empty or not.

Proof of Claim 1: Suppose $\tilde{C}_\delta \neq \emptyset$ and let $x \in \tilde{C}_\delta \subset C_\delta$. Fix any $\bar{x} \in \partial C \cap K$ such that $\|x - \bar{x}\| \leq \delta$. As $\delta \leq \frac{\eta}{3} \leq \frac{R}{6}$ and $x \in \frac{3R}{2}\mathbb{B}$, we have that $\bar{x} \in (x + \delta\mathbb{B}) \subset 2R\mathbb{B}$. Further,

$$d_\Delta(\bar{x}) \geq d_\Delta(x) - \|x - \bar{x}\| \geq \eta - \delta > \eta/2.$$

Therefore $\bar{x} \in \tilde{C}$, implying it is not empty. If $\Delta \neq \emptyset$, by definition, $\tilde{\rho} > 0$. If $\Delta = \emptyset$, then $\tilde{C} = \partial C \cap K \cap 2R\mathbb{B}$ is compact and $\tilde{C} \cap \partial K = \emptyset$, so $\tilde{\rho} > 0$. In both cases, as $\tilde{C}_\delta \subset \tilde{C} + \delta\mathbb{B}$, $\tilde{\rho}_\delta \geq \tilde{\rho} - \delta$.

We now define the key constants that the proof of Theorem 1 will require:

$$k_0 := 4/\epsilon \qquad \gamma := \sqrt{1 - \left(\frac{\epsilon}{16M}\right)^2}$$
$$k := \frac{2}{\epsilon}(1 + \gamma) \qquad \tilde{\gamma} := \frac{1 + \gamma}{2}$$

Notice that

$$k(1 - \tilde{\gamma}) = \frac{k(1 - \gamma)}{2} = \frac{\epsilon}{(16M)^2}. \tag{12}$$

Define

$$\delta := \min\left(1, \frac{\eta}{3}, \frac{\tilde{\rho}}{4 + 16k_0 M}, \frac{\epsilon}{8c_F M}, \frac{R}{32kM}\right). \tag{13}$$

As $\delta \leq 1$, we may replace ϵ by $\epsilon\delta$ in (10).

Let us show that for any $x_0 \in (C_\delta \cap K \cap R\mathbb{B})$, we can define, for $j \in \mathbb{N}$, a feasible F-trajectory $y_j(\cdot)$ on $[0, t_j]$ with $t_j > 0$, satisfying the following properties for $d(x) := d_{C \cap K}(x)$

$$x_{j+1} := y_{j+1}(0) = y_j(t_j) \in \frac{3R}{2}\mathbb{B} \qquad \sum_{j=0}^{\infty} t_j \leq 16kd(x_0)$$

and the two inequalities

$$d(x_j) \le \tilde{\gamma}^j d(x_0) \qquad\qquad \|x_j - x_0\| \le \frac{\epsilon}{16M} \sum_{k=0}^{j-1} \tilde{\gamma}^k d(x_0). \tag{14}$$

Let us first make sure that the $(x_j)_j$ satisfying (14) belong to $\frac{3R}{2}\mathbb{B}$, by applying (12). Indeed

$$\|x_j\| \le \|x_0\| + \|x_j - x_0\| \le R + \frac{\epsilon}{16M}\frac{1}{1-\tilde{\gamma}} d(x_0) \le R + 16kM\delta \le \frac{3R}{2}.$$

Let $j \in \mathbb{N}$. The inequalities (14) are obviously satisfied for $j = 0$, so we will proceed by induction on j. Assume we have already constructed our trajectories up to step j and have not yet reached C (i.e. $x_j \notin C$). Let $\bar{x}_j \in \partial(C \cap K)$ be such that $d(x_j) = \|x_j - \bar{x}_j\|$ (i.e. $\bar{x}_j \in \Pi_{C\cap K}(x_j)$).

As $x_j \in C_\delta \cap K \cap (3R/2)\mathbb{B} = \tilde{C}_\delta \cup (\Delta + \eta\mathbb{B})$, we distinguish two cases. In Case 1, we consider the situation where $\Delta \neq \emptyset$ and $x_j \in (\Delta + \eta\mathbb{B})$. If the x_{j+1} that we design below belongs to $\tilde{C}_\delta$, then we move to Case 2, otherwise the induction proceeds according to Case 1. In Case 2, $x_j \in \tilde{C}_\delta$ and we build by induction $(x_m)_{m\ge j+1} \in C_\delta$. Owing to Claim 1, we will show that, for any $m \ge j+1$ and $t \in [0, t_m]$, the designed trajectory $y_m(\cdot)$ satisfies $d_{\partial K}(y_m(t)) > 0$. This latter property ensures that once the designed trajectory is far enough from ∂K (i.e. $x_j \in \tilde{C}_\delta$), it stays so, and we can focus on reaching C.

Case 1: Suppose $\Delta \neq \emptyset$ and $x_j \in (\Delta + \eta\mathbb{B})\backslash C$, then take any $\tilde{x}_j \in \Delta \cap \mathbb{B}(x_j, \eta)$. Consider $\tilde{v}_j \in \operatorname{co} F(\tilde{x}_j)$ satisfying (9). As $\tilde{x}_j \in \Delta \subset C \cap K \cap 2R\mathbb{B}$,

$$\begin{gathered}\|x_j - \bar{x}_j\| = d(x_j) \le \|x_j - \tilde{x}_j\| \le \eta \\ \|\tilde{x}_j - \bar{x}_j\| \le \|\tilde{x}_j - x_j\| + \|x_j - \bar{x}_j\| \le 2\eta \le \min(\eta_1, \eta_0) \\ \|\bar{x}_j\| \le \|x_j\| + \|x_j - \bar{x}_j\| \le \frac{3R}{2} + \eta \le 2R\end{gathered}$$

we may thus apply (11) at $\tilde{x}_j$ and $\bar{x}_j$ (instead of $\tilde{x}$ and x) and then at $\bar{x}_j$ and x_j. In this way, we get $\bar{v}_j \in \operatorname{co} F(\bar{x}_j)$ and $v_j \in \operatorname{co} F(x_j)$ both satisfying (11)

$$\|\tilde{v}_j - \bar{v}_j\| \le \epsilon/4 \qquad\qquad \|v_j - \bar{v}_j\| \le \epsilon/4$$

and the second (resp. first) line of (9) at $\tilde{x}_j$ and $\bar{x}_j$ (resp. at $\tilde{x}_j$ and x_j)

$$\bar{x}_j + [0,\epsilon](\tilde{v}_j + \epsilon\mathbb{B}) \subset C \cap K \qquad\qquad x_j + [0,\epsilon](\tilde{v}_j + \epsilon\mathbb{B}) \subset K$$

which implies that

$$\bar{x}_j + [0,\epsilon](v_j + \epsilon/2\mathbb{B}) \subset C \cap K \qquad\qquad x_j + [0,\epsilon](v_j + \epsilon/2\mathbb{B}) \subset K.$$

Using that $(x_j - \bar{x}_j) \in N_{C\cap K}(\bar{x}_j)\backslash\{0\}$, relation (3) gives $\langle v_j, x_j - \bar{x}_j\rangle \le -\|x_j - \bar{x}_j\|\epsilon/2$. Hence we have shown the two following formulas relating x_j and v_j:

$$\langle v_j, x_j - \bar{x}_j\rangle \le -\|x_j - \bar{x}_j\|\epsilon/2 \qquad\qquad x_j + [0,\epsilon](v_j + \epsilon/2\mathbb{B}) \subset K \tag{15}$$

Let us now design a trajectory starting at x_j. Define the duration $t_j > 0$ as follows

$$t_j := \frac{\epsilon}{16M^2} d(x_j) \tag{16}$$

By [1, Theorem 9.5.3], there exists a c_F-Lipschitz selection f from the set-valued map $\operatorname{co} F$ defined on $2R\mathbb{B}$, satisfying $v_j = f(x_j)$. Let $\check{y}_j(\cdot)$ be the unique solution of the differential equation $x'(t) = f(x(t))$ on $[0, t_j]$ with initial condition $x(0) = x_j$. Let $w_j(\cdot) := f(\check{y}_j(\cdot))$ on $[0, t_j]$. Then

$$\check{y}_j(t) = x_j + \int_0^t w_j(s) ds \tag{17}$$

Furthermore we have $\|w_j(s) - v_j\| \leq c_F \|\check{y}_j(s) - x_j\|$, and

$$\|\check{y}_j(t) - x_j\| \leq \|tv_j\| + \int_0^t \|w_j(s) - v_j\| \leq Mt_j + c_F \int_0^t \|\check{y}_j(s) - x_j\|$$

We apply Gronwall's lemma as $\check{y}_j(t)$ is continuous and we take into account (10), recalling that $\delta \leq 1$

$$\|\check{y}_j(t) - x_j\| \leq Mt_j e^{c_F t_j} \leq \frac{\epsilon\delta}{16M} e^{\epsilon\delta c_F/(16M^2)} \leq R/2 \tag{18}$$

Therefore $\check{y}_j(t) \in 2R\mathbb{B}$, which implies that $w_j(t) \in M\mathbb{B}$. Thus (17) gives $\|\check{y}_j(s) - x_j\| \leq Mt_j$. As $M \geq 1$ and $\epsilon \leq 1$:

$$\begin{aligned}
\|\check{y}_j(t) - x_j - tv_j\| &\leq \int_0^t \|w_j(s) - v_j\| \leq c_F \int_0^t \|\check{y}_j(s) - x_j\| \leq c_F M t_j \cdot t \\
&\leq c_F M \frac{\epsilon\delta}{16M^2} \cdot t \leq c_F M \frac{\epsilon}{16M^2} \frac{\epsilon}{8c_F M} \cdot t \leq \frac{\epsilon}{2} \cdot t
\end{aligned}$$

Thanks to (15), this ensures that $\check{y}_j(t) \in (x_j + t(v_j + \epsilon/2\mathbb{B})) \subset K$. We have so far designed a feasible $\operatorname{co} F$-trajectory. Applying (15), we obtain furthermore for any $t \in [0, t_j]$:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\check{y}_j(t) - \bar{x}_j\|^2 &= \langle w_j(t), \check{y}_j(t) - \bar{x}_j \rangle \\
&= \langle v_j, x_j - \bar{x}_j \rangle + \langle w_j(t), \check{y}_j(t) - x_j \rangle + \langle w_j(t) - v_j, x_j - \bar{x}_j \rangle \\
&\leq -\frac{\epsilon}{2}\|x_j - \bar{x}_j\| + M\|\check{y}_j(t) - x_j\| + c_F\|\check{y}_j(t) - x_j\|\|x_j - \bar{x}_j\| \\
&\leq -\frac{\epsilon}{2} d(x_j) + (M + c_F d(x_j))Mt \\
&\leq \left(-\frac{\epsilon}{2} + c_F M t_j\right) d(x_j) + M^2 t_j \\
&= \left(-\frac{\epsilon}{2} + \frac{c_F\delta}{M}\frac{\epsilon}{16} + \frac{\epsilon}{16}\right) d(x_j) \leq -\frac{3\epsilon}{8} d(x_j) \leq -\frac{\epsilon}{32} d(x_j) = -\frac{M^2 t_j}{2}
\end{aligned}$$

Consequently, by integration:

$$d^2(\check{y}_j(t_j)) \leq \|\check{y}_j(t_j) - \bar{x}_j\|^2 \leq d^2(x_j) - M^2 t_j^2 = \left(1 - M^2\left(\frac{\epsilon}{16M^2}\right)^2\right) d^2(x_j) = \gamma^2 d^2(x_j)$$

However we cannot set $\check{y}_j(t_j)$ as the next x_{j+1}, as it is only a $\operatorname{co} F$-trajectory for the time being. In order to apply the relaxation theorem, we need a globally Lipschitz continuous set-valued map. Fix any $\epsilon_j > 0$. Let $\tilde{F}$ be defined in any $x \in \mathbb{R}^n$ as $\tilde{F}(x) := F(\Pi_{(2R+\epsilon_j)\mathbb{B}}(x))$ where $\Pi_{(2R+\epsilon_j)\mathbb{B}}(x)$ is the unique projection of x into $(2R+\epsilon_j)\mathbb{B}$. Since the projection on a ball is Lipschitz, we deduce that $\tilde{F}$ is globally Lipschitz. As $\check{y}_j([0,t_j]) \subset 2R\mathbb{B}$, $\check{y}_j(\cdot)$ is also an $\tilde{F}$-trajectory. Thanks to the relaxation theorem, we may thus build an $\tilde{F}$-trajectory $\hat{y}_j(\cdot)$ starting from x_j and enjoying the following property:

$$\|\hat{y}_j - \check{y}_j\|_{L_\infty([0,t_j])} \leq \epsilon_j$$

As $\check{y}_j([0,t_j]) \subset 2R\mathbb{B}$, $\hat{y}_j([0,t_j]) \subset (2R+\epsilon_j)\mathbb{B}$, on which $\tilde{F}$ and F coincide. Hence $\hat{y}_j(\cdot)$ is an F-trajectory.

If $\hat{y}_j([0,t_j]) \subset K$, then we keep it as our feasible F-trajectory. Otherwise, if it leaves K even during a short time, we correct it into an F-feasible trajectory $y_j(\cdot)$ staying in $\operatorname{Int} K$, through [3, Theorem 2.3] which we apply on $2R\mathbb{B}$ and time interval $[0,t_j] \subset [0,t_{max}]$ with $t_{max} = \epsilon\delta/(16M^2) \geq t_j$. This implies the existence of a constant $L \geq 1$ (depending only on R and F) with the following property (as $\check{y}_j([0,t_j]) \subset K$):

$$\begin{gathered}\|y_j - \hat{y}_j\|_{L_\infty([0,t_j])} \leq L\|d_K(\hat{y}_j(\cdot))\|_{L_\infty([0,t_j])} \leq L\|\hat{y}_j - \check{y}_j\|_{L_\infty([0,t_j])} \leq L\epsilon_j \\ d(y_j(t_j)) \leq d(\hat{y}_j(t_j)) + \|y_j(t_j) - \hat{y}_j(t_j)\| \leq \gamma d(x_j) + L\epsilon_j\end{gathered}$$

The above relations remain true even if $\hat{y}_j([0,t_j]) \subset K$. We set $\epsilon_j := \frac{1-\gamma}{2L} d(x_j) \leq \delta$ and $x_{j+1} := y_j(t_j)$, thus:

$$d(x_{j+1}) \leq (\gamma + \frac{1-\gamma}{2}) d(x_j) = \tilde{\gamma} d(x_j)$$

Moreover we derive from (14) and (16):

$$\|x_{j+1} - x_0\| \leq \|x_j - x_0\| + \|x_{j+1} - x_j\| \leq \frac{\epsilon}{16M} \sum_{k=0}^{j-1} \tilde{\gamma}^k d(x_0) + Mt_j \leq \frac{\epsilon}{16M} \sum_{k=0}^{j} \tilde{\gamma}^k d(x_0)$$

If $x_{j+1} \in (\Delta + \eta\mathbb{B}) \backslash C$, then we remain in Case 1. Otherwise, if $x_{j+1} \in \tilde{C}_\delta \backslash C$, we move to Case 2.

Case 2: Suppose that $x_j \in \tilde{C}_\delta \backslash C$. The trajectory construction is similar to Case 1 and even simpler as we do not have to consider the point $\tilde{x}_j$. As $\eta \geq \delta$, we can still select $\bar{v}_j \in \operatorname{co} F(\bar{x}_j)$ and $v_j \in \operatorname{co} F(x_j)$ satisfying (8) and $\|v_j - \bar{v}_j\| \leq \epsilon/4$. We define t_j and $\check{y}_j(\cdot)$ as in (16), and apply (18). This leads to the same computations for $\frac{d}{dt}\|\check{y}_j(t) - \bar{x}_j\|^2$ and for $d(\check{y}_j(t_j))$. We define again through relaxation an F-trajectory $\hat{y}_j(\cdot)$ with the same ϵ_j and we set $x_{j+1} = \hat{y}_j(t_j) \in C_\delta$, which satisfies (14). Then repeating the above steps, we build a sequence $(x_m)_{m \geq j+1} \in C_\delta$ connected by trajectories $\hat{y}_m(\cdot)$.

We no longer have to check the feasibility of such $\hat{y}_m(\cdot)$. As a matter of fact, recall that owing to Claim 1, as $\tilde{C}_\delta \neq \emptyset$, $\tilde{\rho}$ is finite. Let $m \geq j+1$. Using that $\gamma \leq \tilde{\gamma} \leq 1$, $k \leq k_0$ and

$d(x_j) \le \delta \le \tilde{\rho}/(4+16k_0M)$, we deduce

$$\begin{aligned}
d_{\partial K}(\hat{y}_m(t)) &\ge d_{\partial K}(x_j) - \|\hat{y}_m(t) - x_j\| \\
&\ge \tilde{\rho}_\delta - \|\hat{y}_m(t) - \check{y}_m(t)\| - \|\check{y}_m(t) - \bar{x}_m\| - \|\bar{x}_m - x_m\| - \|x_m - x_j\| \\
&\ge \tilde{\rho} - \delta - L\epsilon_m - 2d(x_m) - \frac{\epsilon}{16M(1-\tilde{\gamma})} d(x_j) \\
&\ge \tilde{\rho} - \delta - \frac{1-\gamma}{2} d(x_m) - 2\tilde{\gamma}^{m-j} d(x_j) - 16kMd(x_j) \\
&\ge \tilde{\rho} - \delta - \frac{1}{2}\tilde{\gamma}^{m-j} d(x_j) - 2d(x_j) - 16kMd(x_j) \\
&\ge \tilde{\rho} - (\frac{7}{2} + 16k_0M)\delta > 0
\end{aligned}$$

The above computation ensures that whenever an x_j is both close enough to C and far enough from ∂K, we can focus only on reaching C.

To conclude, we have built a bounded sequence $(x_j)_{j\ge 0}$ connected by feasible F-trajectories $y_j(\cdot)$, satisfying both (14) and:

$$\lim_{j\to+\infty} d(x_j) = 0$$

Using (12), define τ as follows:

$$\tau := \sum_{j=0}^{\infty} t_j = \frac{\epsilon}{16M^2} \sum_{j=0}^{\infty} d(x_j) \le \frac{\epsilon}{16M^2} \frac{1}{1-\tilde{\gamma}} d(x_0) = 16kd(x_0)$$

Concatenating all the feasible F-trajectories $y_j(\cdot)$, we get a feasible F-trajectory $y(\cdot)$ starting at x_0, defined on $[0,\tau]$ and reaching $C \cap K$ at time τ. Hence assertion (4) follows:

$$\tau_{min}(x_0) \le \tau \le 16kd(x_0)$$

which concludes the proof, replacing $16k$ by k to recover (4).

■

Proof of Theorem 2: Let $x_0 \in \mathrm{Capt}_F(K,C)\backslash C$ and $\xi > 0$ such that $\mathbb{B}(x_0,\xi) \cap C = \emptyset$. Let $y_0(\cdot)$ be a feasible F-trajectory starting at x_0 and reaching C at $\bar{x}_0$ at some time τ_0 where $\tau_{min}(x_0) \le \tau_0 \le 2\tau_{min}(x_0)$. Let $x_1 \in K \cap \mathbb{B}(x_0,\xi)$. Define

$$R := (\|\bar{x}_0\| + 1)\, e^{M(2\tau_{min}(x_0)+1)} \qquad \text{where } M := \sup_{z\in\mathbb{B}, t\in[0,\tau_0]} \ \sup_{v\in F(y_0(t)+z)} \|v\|$$

Let k_F be as in Assumption 2 for this value of R. Let $\tilde{F}$ be defined in any $x \in \mathbb{R}^n$ as $\tilde{F}(x) := F(\Pi_{R\mathbb{B}}(x))$. Since the projection on a ball is Lipschitz with constant 1, we deduce that $\tilde{F}$ is globally Lipschitz with constant k_F. As $y_0([0,\tau_0]) \subset R\mathbb{B}$, $y_0(\cdot)$ is also an $\tilde{F}$-trajectory. We may then apply the Filippov's existence theorem (see e.g. [1, Theorem 10.4.1, p 384]), to design an $\tilde{F}$-trajectory $\hat{y}_1(\cdot)$ on $[0,\tau_0]$ starting from x_1 such that:

$$\|y_0 - \hat{y}_1\|_{L_\infty([0,\tau_0])} \le \|x_0 - x_1\| e^{k_F\tau_0} \le c\cdot\xi \text{ where } c := e^{k_F(2\tau_{min}(x_0)+1)}$$

Take from now on $\xi \leq 1/c$. Therefore, for any $t \in [0, \tau_0]$, $\hat{y}_1(t) \in (y_0(t) + \mathbb{B})$ and in particular $\hat{y}_1(\tau_0) \in (\bar{x}_0 + \mathbb{B}))$, from which we derive that $\hat{y}_1([0, \tau_0]) \subset R\mathbb{B}$, on which $\tilde{F}$ and F coincide. We thus conclude that $\hat{y}_1(\cdot)$ is an F-trajectory. If it stays within K, we keep it. Otherwise we apply [3, Theorem 2.3] to retrieve an F-trajectory $y_1(\cdot)$ on $[0, \tau_0]$ starting from x_1 and staying in K, satisfying in both cases for an $L \geq 0$ (depending only on R and F):

$$\|y_0-y_1\|_{L_\infty([0,\tau_0])} \leq \|y_0-\hat{y}_1\|_{L_\infty([0,\tau_0])}+\|\hat{y}_1-y_1\|_{L_\infty([0,\tau_0])} \leq (1+L)\|y_0-\hat{y}_1\|_{L_\infty([0,\tau_0])} \leq c(1+L)\xi$$

Let $\delta > 0$, $k > 0$ and $d(\cdot)$ be as in (4). We choose ξ such that $c(1 + L)\xi \leq \min(1/k, \delta, 1)$. As $y_1(\tau_0) \in (C + \delta\mathbb{B}) \cap K \cap R\mathbb{B}$, we deduce from the dynamic programming principle that:

$$\begin{aligned} \tau_{min}(x_1) &\leq \tau_0 + \tau_{min}(y_1(\tau_0)) \leq \tau_0 + kd(y_1(\tau_0)) \\ &\leq 2\tau_{min}(x_0) + ck(1 + L)\xi \leq 2\tau_{min}(x_0) + 1 \end{aligned}$$

We have thus shown that $x_1 \in \mathrm{Capt}_F(K, C)$. Since x_1 is an arbitrary point in K in a neighborhood of x_0, $\mathrm{Capt}_F(K, C)$ is open in K.

We now repeat the above strategy for $x_1, x_2 \in \mathbb{B}(x_0, \xi/2) \cap K$. Let $\epsilon_1 \geq 0$. Let $y_1(\cdot)$ be a feasible F-trajectory starting at x_1 and reaching C at $\bar{x}_1$ at time $\tau_1 \leq \tau_{min}(x_1)+\epsilon_1$. Then design a feasible F-trajectory $y_2(\cdot)$ on $[0, \tau_1]$ starting from x_2. With the same arguments as above, $y_2(\tau_1) \in R\mathbb{B}$ and:

$$\begin{aligned} d(y_2(\tau_1)) &\leq \|y_2(\tau_1) - \bar{x}_1\| \leq c(1 + L)\|x_2 - x_1\| \leq c(1 + L)\xi \leq \delta \\ \tau_{min}(x_2) &\leq \tau_1 + \tau_{min}(y_2(\tau_1)) \leq \tau_{min}(x_1) + \epsilon_1 + c(1 + L)k\|x_2 - x_1\|. \end{aligned}$$

As the roles of x_1 and x_2 can be permuted and ϵ_1 is arbitrary, $\tau_{min}(\cdot)$ is Lipschitz continuous on $\mathbb{B}(x_0, \xi/2) \cap K$.

■

Proof of Proposition 1: This constructive proof is largely similar to that of Theorem 1, we thus focus only on the few differences as here $K = \mathbb{R}^n$ and $d(x) = \|x - \bar{x}\|$.

Let $R = \|\bar{x}\|+1$ and define the constants k_F, c_F and M accordingly. Fix $\epsilon > 0$ satisfying both $\epsilon\mathbb{B} \subset \mathrm{co}\, F(\bar{x})$ and (10), then define the other constants η_1, k_0, k, γ and $\tilde{\gamma}$ as in the proof of Theorem 1. Set δ as follows

$$\delta := \min\left(1, \frac{\eta_1}{6}, \frac{R}{6}, \frac{\epsilon}{8c_F M}, \frac{R}{32kM}\right).$$

Let $x_0 \in (\bar{x}+\delta\mathbb{B})$, $j \in \mathbb{N}$ and suppose that (14) is satisfied at x_j. Let $\bar{v}_j = -\epsilon(x_j - \bar{x})/\|x_j - \bar{x}\|$. As $\bar{v}_j \in \mathrm{co}\, F(\bar{x})$, through (11), we can fix a $v_j \in \mathrm{co}\, F(x_j)$ such that $\|v_j - \bar{v}_j\| \leq \epsilon/4$. Hence

$$\langle v_j, x_j - \bar{x}\rangle \leq \langle \bar{v}_j, x_j - \bar{x}\rangle + \|x_j - \bar{x}\|\|v_j - \bar{v}_j\| \leq -\frac{\epsilon}{2}\|x_j - \bar{x}\|$$

and we recover (15). Let t_j as in (16), and construct similarly $\hat{y}_j(\cdot)$ through (17) and relaxation. This defines the next point $x_{j+1} = \hat{y}_j(t_j)$ for the induction. In conclusion, the bounded sequence $(x_j)_{j\geq 0}$ converges to $\bar{x}$, which is reached in time less than $16k\|x_0 - \bar{x}\|$ by an F-trajectory starting at x_0.

■

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