Characterizing order isomorphisms of sup-stable function spaces: continuous, Lipschitz, c-convex, and beyond.. via inf/sup irreducibility

Pierre-Cyril Aubin-Frankowski

Postdoc at TU Wien - VADOR

AKOR Seminar, TU Wien, April 11th 2024
Joint work with Stéphane Gaubert (INRIA, France)

Why should one be interested in characterizing transformations?

1. nonlinear versions of the Banach-Stone theorem, used to study morphisms between sets in [Weaver, 1994, Leung and Tang, 2016]. As a reminder, linear Banach-Stone:

   Every linear surjective isometry on $C(\mathcal{X}, \mathbb{R})$ *is of the form* $(Jf)(x) = g(x) \cdot f(\phi(x))$;

Why should one be interested in characterizing transformations?

1. nonlinear versions of the Banach-Stone theorem, used to study morphisms between sets in [Weaver, 1994, Leung and Tang, 2016]. As a reminder, linear Banach-Stone:

   Every linear surjective isometry on $C(\mathcal{X},\mathbb{R})$ *is of the form* $(Jf)(x) = g(x)\cdot f(\phi(x))$*;*

2. some transformations are quite exceptional, such as the Fenchel transform over convex l.s.c. functions [Artstein-Avidan and Milman, 2009]

   Every order-reversing isomorphism on $Cvx(\mathbb{R}^d,\mathbb{R})$ *is of the form*
   $$(Tf)(x) = \langle c, x\rangle + \delta + f^*(Ax+c);$$

Why should one be interested in characterizing transformations?

1. nonlinear versions of the Banach-Stone theorem, used to study morphisms between sets in [Weaver, 1994, Leung and Tang, 2016]. As a reminder, linear Banach-Stone:

   Every linear surjective isometry on $C(\mathcal{X}, \mathbb{R})$ is of the form $(Jf)(x) = g(x) \cdot f(\phi(x))$;

2. some transformations are quite exceptional, such as the Fenchel transform over convex l.s.c. functions [Artstein-Avidan and Milman, 2009]

   Every order-reversing isomorphism on $Cvx(\mathbb{R}^d, \mathbb{R})$ is of the form
   $$(Tf)(x) = \langle c, x\rangle + \delta + f^*(Ax + c);$$

3. order-reversing involutions are symmetries for some equations like the complex Monge-Ampère [Lempert, 2017, Berndtsson et al., 2020] and provide Blaschke-Santaló type inequalities [Artstein-Avidan et al., 2023]

Why should one be interested in characterizing transformations?

1. nonlinear versions of the Banach-Stone theorem, used to study morphisms between sets in [Weaver, 1994, Leung and Tang, 2016]. As a reminder, linear Banach-Stone:

   Every linear surjective isometry on $C(\mathcal{X}, \mathbb{R})$ *is of the form* $(Jf)(x) = g(x) \cdot f(\phi(x))$*;*

2. some transformations are quite exceptional, such as the Fenchel transform over convex l.s.c. functions [Artstein-Avidan and Milman, 2009]

   Every order-reversing isomorphism on $Cvx(\mathbb{R}^d, \mathbb{R})$ *is of the form*
   $$(Tf)(x) = \langle c, x\rangle + \delta + f^*(Ax + c);$$

3. order-reversing involutions are symmetries for some equations like the complex Monge-Ampère [Lempert, 2017, Berndtsson et al., 2020] and provide Blaschke-Santaló type inequalities [Artstein-Avidan et al., 2023]

The main idea: finding subsets invariant under J

Linear Banach-Stone on compact $\mathcal{X}$:

Every linear surjective isometry on $C(\mathcal{X}, \mathbb{R})$ is of the form $(Jf)(x) = g(x) \cdot f(\phi(x))$.

The proof mainly consists in showing that the adjoint J^* maps the extreme points of the dual ball on themselves, which are the Dirac masses $\pm\delta_x$.

Then $(Jf)(x) = \langle \delta_x, Jf \rangle = \langle J^*\delta_x, f \rangle = \langle g(x)\delta_{\phi(x)}, f \rangle = g(x) \cdot f(\phi(x))$

What should be the analogue of the δ_x? What is "extremality" in our setting?

Theorem (Main theorem on (max,+)-isomorphisms)

Let $\mathcal{X}$ and $\mathcal{Y}$ be sets. Let $\mathcal{G}$ (resp. $\mathcal{F}$) be a subset of $\overline{\mathbb{R}}^{\mathcal{X}}$ (resp. $\overline{\mathbb{R}}^{\mathcal{Y}}$), with both $\mathcal{F}$ and $\mathcal{G}$ being proper and separating, stable by arbitrary suprema and addition of scalars. Define

$$e_x = \sup_{u \in \mathcal{G}} u(\cdot) - u(x), \quad e'_y = \sup_{v \in \mathcal{F}} v(\cdot) - v(y)$$

Let J be a (max,+)-isomorphism from $\mathcal{F}$ onto $\mathcal{G}$, then the following statements are equivalent:

1. *for all $y \in \mathcal{Y}$, $J(e'_y) \in \{e_x + \lambda\}_{x \in \mathcal{X}, \lambda \in \overline{\mathbb{R}}}$;*
2. *there exists $g : \mathcal{X} \to \mathbb{R}$ and a bijective $\phi : \mathcal{X} \to \mathcal{Y}$ such that*

$$Jf(x) = g(x) + f(\phi(x)) \tag{1}$$

and for all $f \in \mathcal{F}$, $h \in \mathcal{G}$, $(g + f \circ \phi) \in \mathcal{G}$ and $(-g \circ \phi^{-1} + h \circ \phi^{-1}) \in \mathcal{F}$.

We will show that 1) actually holds for many sets. We also study the more general order isomorphisms for some sets of functions (Lipschitz, convex, l.s.c.).

For many sets, every (max,+)-isomorphism is of the form:

$$Jf(x) = g(x) + f(\phi(x))$$

Set $\mathcal{X}$	Function space $\mathcal{G}$	Translation g	Reparametrization ϕ
Hausdorff topological space	l.s.c. functions	continuous	homeomorphism
complete metric space	1-Lipschitz functions	constant	isometry
complete metric space	Lipschitz functions	Lipschitz	bi-Lipschitz homeomorphism
locally convex Hausdorff topological	l.s.c. convex functions	continuous affine	continuous affine

Let $\overline{\mathbb{R}} = [-\infty, +\infty]$, fix a set $\mathcal{X}$ and $\mathcal{G} \subset \overline{\mathbb{R}}^{\mathcal{X}}$

- $\mathcal{G}$ is sup-stable if $\sup_{\alpha \in \mathcal{A}} g_\alpha \in \mathcal{G}$ for any $(h_\alpha)_{\alpha \in \mathcal{A}} \subset \mathcal{G}$
- $\mathcal{G}$ is a complete subspace of $\overline{\mathbb{R}}^{\mathcal{X}}$ if $\mathcal{G}$ is sup-stable and $(g + \lambda) \in \mathcal{G}$ for $g \in \mathcal{G}$ and $\lambda \in \overline{\mathbb{R}}$
- the sup-closure of $\mathcal{G}$ is $\overline{\mathcal{G}}^{\sup} := \{\sup_{\alpha \in \mathcal{A}} h_\alpha \mid \mathcal{A}$ an index set, $\{h_\alpha\}_{\alpha \in \mathcal{A}} \subset \mathcal{G}\}$
- the inf-closure of $\mathcal{G}$ is $\overline{\mathcal{G}}^{\inf} := \{\inf_{\alpha \in \mathcal{A}} h_\alpha \mid \mathcal{A}$ an index set, $\{h_\alpha\}_{\alpha \in \mathcal{A}} \subset \mathcal{G}\}$
- the infimum relatively to $\mathcal{G}$ of a family $(g_\alpha)_{\alpha \in \mathcal{A}} \in \mathcal{G}^{\mathcal{A}}$ is

$$\inf_\alpha^{\mathcal{G}} g_\alpha := \max\{h \in \mathcal{G} \mid \forall \alpha \in \mathcal{A},\ h \leq g_\alpha\}, \quad \inf^{\mathcal{G}} g := \max\{h \in \mathcal{G} \mid h \leq g\}$$

Let $\overline{\mathbb{R}} = [-\infty, +\infty]$, fix a set $\mathcal{X}$ and $\mathcal{G} \subset \overline{\mathbb{R}}^{\mathcal{X}}$

- $\mathcal{G}$ is sup-stable if $\sup_{\alpha \in \mathcal{A}} g_\alpha \in \mathcal{G}$ for any $(h_\alpha)_{\alpha \in \mathcal{A}} \subset \mathcal{G}$
- $\mathcal{G}$ is a complete subspace of $\overline{\mathbb{R}}^{\mathcal{X}}$ if $\mathcal{G}$ is sup-stable and $(g + \lambda) \in \mathcal{G}$ for $g \in \mathcal{G}$ and $\lambda \in \overline{\mathbb{R}}$
- the sup-closure of $\mathcal{G}$ is $\overline{\mathcal{G}}^{\text{sup}} := \{\sup_{\alpha \in \mathcal{A}} h_\alpha | \, \mathcal{A} \text{ an index set}, \{h_\alpha\}_{\alpha \in \mathcal{A}} \subset \mathcal{G}\}$
- the inf-closure of $\mathcal{G}$ is $\overline{\mathcal{G}}^{\text{inf}} := \{\inf_{\alpha \in \mathcal{A}} h_\alpha | \, \mathcal{A} \text{ an index set}, \{h_\alpha\}_{\alpha \in \mathcal{A}} \subset \mathcal{G}\}$
- the infimum relatively to $\mathcal{G}$ of a family $(g_\alpha)_{\alpha \in \mathcal{A}} \in \mathcal{G}^{\mathcal{A}}$ is

$$\inf_\alpha^{\mathcal{G}} g_\alpha := \max\{h \in \mathcal{G} \,|\, \forall \alpha \in \mathcal{A}, \, h \leq g_\alpha\}, \quad \inf^{\mathcal{G}} g := \max\{h \in \mathcal{G} \,|\, h \leq g\}$$

- $f \in \mathcal{G}$ is sup-irreducible if, for all $g, h \in \mathcal{G}$, $f = \sup(g, h) \implies f = g$ or $f = h$
- $f \in \mathcal{G}$ is inf-irreducible if, for all $g, h \in \mathcal{G}$, $f = \inf(g, h) \implies f = g$ or $f = h$
- $f \in \mathcal{G}$ is $\mathcal{G}$-relatively-inf-irreducible if, for all $g, h \in \mathcal{G}$, $f = \inf^{\mathcal{G}}(g, h) \implies f = g$ or $f = h$

Remark: $\mathcal{G}$-relatively-inf-irreducible functions are inf-irreducible (converse is false). Set

$$\delta_x^{\perp}(y) := \begin{cases} 0 & \text{if } y = x, \\ -\infty & \text{otherwise}, \end{cases} \quad \delta_x^{\top}(y) := \begin{cases} 0 & \text{if } y = x, \\ +\infty & \text{otherwise}. \end{cases} \tag{2}$$

If $\delta_x^{\top} \in \mathcal{G}$, then it is $\mathcal{G}$-relatively-inf-irreducible, and if $\delta_x^{\perp} \in \mathcal{G}$, then it is sup-irreducible.

A map $J : \mathcal{F} \to \mathcal{G}$ where $\mathcal{F}$ and $\mathcal{G}$ are partially ordered sets is (isoφ=isomorphism)

- an order isoφ if it is invertible and if this map and its inverse are both order preserving, i.e. for all $f, g \in \mathcal{F}$, $f \geq g \Leftrightarrow Jf \geq Jg$
- a max-isoφ if it is invertible and if it commutes with suprema, i.e. $J(\sup(f, g)) = \sup(Jf, Jg)$, assuming that $\mathcal{G}$ and $\mathcal{F}$ are sup-stable;
- a (max,+)-isoφ if it is a max-isoφ and if we have $J(f + \lambda) = Jf + \lambda$ for $\lambda \in \mathbb{R}$, assuming that $\mathcal{G}$ and $\mathcal{F}$ are complete subspaces of $\overline{\mathbb{R}}^{\mathcal{X}}$;

A map $J : \mathcal{F} \to \mathcal{G}$ where $\mathcal{F}$ and $\mathcal{G}$ are partially ordered sets is (isoφ=isomorphism)

- an order isoφ if it is invertible and if this map and its inverse are both order preserving, i.e. for all $f, g \in \mathcal{F}$, $f \geq g \Leftrightarrow Jf \geq Jg$
- a max-isoφ if it is invertible and if it commutes with suprema, i.e. $J(\sup(f, g)) = \sup(Jf, Jg)$, assuming that $\mathcal{G}$ and $\mathcal{F}$ are sup-stable;
- a (max,+)-isoφ if it is a max-isoφ and if we have $J(f + \lambda) = Jf + \lambda$ for $\lambda \in \mathbb{R}$, assuming that $\mathcal{G}$ and $\mathcal{F}$ are complete subspaces of $\overline{\mathbb{R}}^{\mathcal{X}}$;
- order reversing if for all $f, g \in \mathcal{F}$, $f \geq g \implies Jf \leq Jg$
- an order anti-isoφ if it is invertible and if this map and its inverse are both order reversing;
- an anti-involution if $J : \mathcal{G} \to \mathcal{G}$, $JJ = \mathrm{Id}_{\mathcal{G}}$ and J is order reversing.

Remark: Order isoφ are a more general notion than max-isoφ, but the two coincide when $\mathcal{F}$ and $\mathcal{G}$ are sup-stable.

Some (max,+) concepts

Definition

A map $B : \overline{\mathbb{R}}^{\mathcal{Y}} \to \overline{\mathbb{R}}^{\mathcal{X}}$ is said to be $\overline{\mathbb{R}}_{\max}$-sesquilinear if $B(\inf\{f_i\}_{i\in I}) = \sup\{Bf_i\}_{i\in I}$ and $B(f+\lambda) = Bf - \lambda$, for any finite index set I and $\lambda \in \overline{\mathbb{R}}$; B is *continuous* if I can be taken infinite. The range of B is $\mathrm{Rg}(B) := \{g \in \overline{\mathbb{R}}^{\mathcal{X}} \mid \exists f \in \overline{\mathbb{R}}^{\mathcal{Y}}, g = Bf\}$.

Proposition (Theorem 3.1, [Singer, 1984])

A map $\bar{B} : \overline{\mathbb{R}}^{\mathcal{Y}} \to \overline{\mathbb{R}}^{\mathcal{X}}$ is $\overline{\mathbb{R}}_{\max}$-sesquilinear and continuous if and only if there exists a kernel $b : \mathcal{X} \times \mathcal{Y} \to \overline{\mathbb{R}}$ such that $\bar{B}f(x) = \sup_{y\in\mathcal{Y}} b(x,y) - f(y)$. Moreover in this case b is uniquely determined by $\bar{B}$ as $b(\cdot,\bar{y}) = \bar{B}\delta^{\top}_{\bar{y}}$.

$$\mathrm{Rg}(B) = \{\sup_{y\in\mathcal{Y}} a_y + b(\cdot,y) \mid a_y \in \mathbb{R}_{\perp}\}. \tag{3}$$

Let $\bar{B} : \overline{\mathbb{R}}^{\mathcal{Y}} \to \overline{\mathbb{R}}^{\mathcal{X}}$ and its transpose $\bar{B}^\circ : \overline{\mathbb{R}}^{\mathcal{X}} \to \overline{\mathbb{R}}^{\mathcal{Y}}$ be defined by

$$\bar{B}f(\cdot) := \sup_{y\in\mathcal{Y}} b(\cdot, y) - f(y), \quad \bar{B}^\circ h(\cdot) := \sup_{x\in\mathcal{X}} b(x, \cdot) - h(x), \forall f \in \overline{\mathbb{R}}^{\mathcal{Y}}, h \in \overline{\mathbb{R}}^{\mathcal{X}} \tag{4}$$

The key relation is that $\bar{B} = \bar{B}\bar{B}^\circ\bar{B}$, see e.g. [Akian et al., 2005]. So $\bar{B}$ and $\bar{B}^\circ$ are anti-isoφ!

Let $\bar{B} : \overline{\mathbb{R}}^{\mathcal{Y}} \to \overline{\mathbb{R}}^{\mathcal{X}}$ and its transpose $\bar{B}^{\circ} : \overline{\mathbb{R}}^{\mathcal{X}} \to \overline{\mathbb{R}}^{\mathcal{Y}}$ be defined by

$$\bar{B}f(\cdot) := \sup_{y\in\mathcal{Y}} b(\cdot, y) - f(y), \quad \bar{B}^{\circ}h(\cdot) := \sup_{x\in\mathcal{X}} b(x, \cdot) - h(x), \forall f \in \overline{\mathbb{R}}^{\mathcal{Y}}, h \in \overline{\mathbb{R}}^{\mathcal{X}} \tag{4}$$

The key relation is that $\bar{B} = \bar{B}\bar{B}^{\circ}\bar{B}$, see e.g. [Akian et al., 2005]. So $\bar{B}$ and $\bar{B}^{\circ}$ are anti-isoφ!

Examples of $b(x, y)$ **and** $\mathrm{Rg}(B)$ **[Singer, 1997]** (adding the constant functions $\pm\infty$):

- For $\mathcal{X} = \mathcal{Y} = \mathbb{R}^N$, $b(x, y) = (x, y)_2$ gives $\mathrm{Rg}(B)$ is the set of proper convex l.s.c. functions.
- For $\mathcal{X} = \mathcal{Y} = \mathbb{R}^N$, $b(x, y) = -\|x - y\|^2$ gives $\mathrm{Rg}(B)$ is the set of proper 1-semiconvex l.s.c. functions, i.e. $f + \|\cdot\|^2$ is convex.

Let $\bar{B} : \overline{\mathbb{R}}^{\mathcal{Y}} \to \overline{\mathbb{R}}^{\mathcal{X}}$ and its transpose $\bar{B}^{\circ} : \overline{\mathbb{R}}^{\mathcal{X}} \to \overline{\mathbb{R}}^{\mathcal{Y}}$ be defined by

$$\bar{B}f(\cdot) := \sup_{y \in \mathcal{Y}} b(\cdot, y) - f(y), \quad \bar{B}^{\circ}h(\cdot) := \sup_{x \in \mathcal{X}} b(x, \cdot) - h(x), \forall f \in \overline{\mathbb{R}}^{\mathcal{Y}}, h \in \overline{\mathbb{R}}^{\mathcal{X}} \tag{4}$$

The key relation is that $\bar{B} = \bar{B}\bar{B}^{\circ}\bar{B}$, see e.g. [Akian et al., 2005]. So $\bar{B}$ and $\bar{B}^{\circ}$ are anti-isoφ!

Examples of $b(x, y)$ **and** $\mathrm{Rg}(B)$ **[Singer, 1997]** (adding the constant functions $\pm\infty$):

- For $\mathcal{X} = \mathcal{Y} = \mathbb{R}^N$, $b(x, y) = (x, y)_2$ gives $\mathrm{Rg}(B)$ is the set of proper convex l.s.c. functions.
- For $\mathcal{X} = \mathcal{Y} = \mathbb{R}^N$, $b(x, y) = -\|x - y\|^2$ gives $\mathrm{Rg}(B)$ is the set of proper 1-semiconvex l.s.c. functions, i.e. $f + \|\cdot\|^2$ is convex.
- For any $\mathcal{X}$ and $\alpha \geq 0$, $b(x, y) = \begin{cases} 0 & \text{if } y = x, \\ -\alpha & \text{otherwise,} \end{cases}$ gives $\mathrm{Rg}(B)$ is the set of functions f which difference $f(x) - f(y)$ is smaller than α.
- For $(\mathcal{X}, d)$ a metric space, $b(x, y) = -d(x, y)^p$ gives $\mathrm{Rg}(B)$ is the set of $(1, p)$-Hölder continuous functions w.r.t. the distance d (i.e. $|f(x) - f(y)| \leq 1 \cdot d(x, y)^p$).

Theorem (Connection between kernels and anti-isoφ)

Let $\mathcal{G}$ (resp. $\mathcal{F}$) be a complete subspace of $\overline{\mathbb{R}}^{\mathcal{X}}$ (resp. $\overline{\mathbb{R}}^{\mathcal{Y}}$). Then TFAE:

1. *there exists a kernel $b : \mathcal{X} \times \mathcal{Y} \to \overline{\mathbb{R}}$ such that $\mathcal{G} = \mathrm{Rg}(B)$ and $\mathcal{F} = \mathrm{Rg}(B^{\circ})$;*
2. *there exists an order anti-isoφ $\bar{F} : \mathcal{F} \to \mathcal{G}$ commuting with the addition of scalars, i.e. $\bar{F}(f + \lambda) = \bar{F}f - \lambda$, for any $\lambda \in \mathbb{R}$ and $f \in \mathcal{F}$.*

In this case, $\bar{F}$ can be taken as the restriction of $\bar{B}$ to $\mathrm{Rg}(B)$. Moreover, for $\mathcal{X} = \mathcal{Y}$, there exists $\bar{F}$ an anti-involution over $\mathcal{G}$ iff there exists a symmetric b such that $\mathcal{G} = \mathrm{Rg}(B)$.

Useful trivial lemmas

Lemma

Let $A, B : \mathcal{F} \to \mathcal{G}$ be two order anti-isoφ. Set $J = A^{-1}B$. Then J is an order isoφ over $\mathcal{F}$, and we have that $B = AJ$ and $A = BJ^{-1}$. In particular, if there exists an anti-involution $\bar{F} : \mathcal{G} \to \mathcal{G}$, every anti-involution over $\mathcal{G}$ writes as $\bar{F}J$ with $J : \mathcal{G} \to \mathcal{G}$ an order isoφ satisfying $\bar{F}J\bar{F}J = \mathrm{Id}_{\mathcal{G}}$.

It is enough to study the order isoφ rather than the more arduous order anti-isoφ!

Lemma

Every order isoφ $J : \mathcal{F} \to \mathcal{G}$ over sets $\mathcal{F} \subset \overline{\mathbb{R}}^{\mathcal{Y}}$ and $\mathcal{G} \subset \overline{\mathbb{R}}^{\mathcal{X}}$ sends sup-irreducible elements (resp. $\mathcal{F}$-relatively-inf-irreducible) of $\mathcal{F}$ onto sup-irreducible elements (resp. $\mathcal{G}$-relatively-inf-irreducible) of $\mathcal{G}$. Every anti-involution T over $\mathcal{G}$ sends $\mathcal{G}$-relatively-inf-irreducible elements of $\mathcal{G}$ onto sup-irreducible elements of $\mathcal{G}$.

Useful trivial lemmas (cont.)

Lemma

Let $J : \mathcal{F} \to \mathcal{G}$ be an order isoφ between $\mathcal{F} \subset \overline{\mathbb{R}}^{\mathcal{Y}}$ and $\mathcal{G} \subset \overline{\mathbb{R}}^{\mathcal{X}}$, such that $\mathcal{F}$ (resp. $\mathcal{G}$) is pointwise dense in the sup-closure $\overline{\mathcal{F}}^{\mathrm{sup}}$ of $\mathcal{F}$ (resp. $\overline{\mathcal{G}}^{\mathrm{sup}}$). Then J can be extended to an order isoφ between $\overline{\mathcal{F}}^{\mathrm{sup}}$ and $\overline{\mathcal{G}}^{\mathrm{sup}}$.

Define the Archimedean class of a function $f \in \mathcal{G}$ as

$$[f] := \{g \in \mathcal{G} \,|\, \exists \alpha \in \mathbb{R},\ f - \alpha \leq g \leq f + \alpha\} \tag{5}$$

Let us put an order on Archimedean classes, saying that $[f] \leq [g]$ if there exists $\alpha \in \mathbb{R}$ such that $f \leq g + \alpha$. A class $[f]$ is maximal if $[f] \leq [g] \implies [f] \geq [g]$.

Lemma

Let $\mathcal{G}$ (resp. $\mathcal{F}$) be a complete subspace of $\overline{\mathbb{R}}^{\mathcal{X}}$ (resp. $\overline{\mathbb{R}}^{\mathcal{Y}}$). Let J be a (max,+)-isoφ from $\mathcal{F}$ onto $\mathcal{G}$. If $f \in \mathcal{F}$ is such that $[f]$ is maximal, then $[Jf]$ is also maximal.

The e_x, the Dirac-like inf-irreducible functions of $\mathcal{G}$

Let $\mathcal{G}$ be a complete subspace of $\overline{\mathbb{R}}^{\mathcal{X}}$. Define, for any $x \in \mathcal{X}$, the function $e_x : \mathcal{X} \to \overline{\mathbb{R}}$ by

$$e_x(\cdot) := \sup\{u \in \mathcal{G} \mid u(x) \leq 0\}. \tag{6}$$

Then $e_x \in \mathcal{G}$, $e_x(x) = 0$ and e_x is inf-irreducible in $\mathcal{G}$. We also have

$$e_x(y) = \sup_{u \in \mathcal{G}} u(y) - u(x), \tag{7}$$

with $-\infty$ absorbing. Moreover, for any $f \in \mathcal{G}$, we have the representation (with $+\infty$ absorbing)

$$f = \inf_{x \in \mathrm{Dom}(f)} e_x + f(x), \quad \text{and} \quad \overline{\mathcal{G}}^{\inf} = \{\inf_x e_x(\cdot) + w_x \mid w_x \in \overline{\mathbb{R}}\}. \tag{8}$$

If $f \in \mathcal{G}$ is such that $[f]$ is maximal, then for all $x_0 \in \mathrm{Dom}(f)$ we have $[e_{x_0}] = [f]$, i.e. we can fix $\lambda_0 \in \mathbb{R}$, such that $e_{x_0} + \lambda_0 \leq f$.

Technical assumption to have $e_x \neq e_{x'} + \lambda$:

The set $\mathcal{G} \subset \overline{\mathbb{R}}^{\mathcal{X}}$ is *proper and point separating* if for any $x, x' \in \mathcal{X}$ with $x \neq x'$, there exists $g_1, g_2 \in \mathcal{G}$ such that $g_1(x), g_2(x), g_1(x'), g_2(x') \in \mathbb{R}$ and $g_1(x) - g_1(x') \neq g_2(x) - g_2(x')$.

Theorem (Main theorem on (max,+)-isoφ)

Let $\mathcal{X}$ and $\mathcal{Y}$ be sets. Let $\mathcal{G}$ (resp. $\mathcal{F}$) be a complete subset of $\overline{\mathbb{R}}^{\mathcal{X}}$ (resp. $\overline{\mathbb{R}}^{\mathcal{Y}}$), with both $\mathcal{F}$ and $\mathcal{G}$ being proper and separating. Set $e_x = \sup_{u \in \mathcal{G}} u(\cdot) - u(x)$. Let $J : \mathcal{F} \to \mathcal{G}$ be a (max,+)-isoφ from $\mathcal{F}$ onto $\mathcal{G}$, then the following statements are equivalent:

1. *for all $y \in \mathcal{Y}$, $J(e'_y) \in \{e_x + \lambda\}_{x \in \mathcal{X},\, \lambda \in \overline{\mathbb{R}}}$;*
2. *there exists $g : \mathcal{X} \to \mathbb{R}$ and a bijective $\phi : \mathcal{X} \to \mathcal{Y}$ such that*
$$Jf(x) = g(x) + f(\phi(x)) \tag{9}$$
and for all $f \in \mathcal{F}$, $h \in \mathcal{G}$, $(g + f \circ \phi) \in \mathcal{G}$ and $(-g \circ \phi^{-1} + h \circ \phi^{-1}) \in \mathcal{F}$.

If $\{\delta_y^\top\}_{y \in \mathcal{Y}} \subset \mathcal{F}$ and $\{\delta_x^\top\}_{x \in \mathcal{X}} \subset \mathcal{G}$, then the statements of Theorem 8 hold for all J!

First application: l.s.c. functions and Lipschitz functions

Let $\mathcal{G}$ (resp. $\mathcal{F}$) be the space of l.s.c. functions over a Hausdorff topological space $\mathcal{X}$ (resp. $\mathcal{Y}$). Then every (max,+)-isoφ J from $\mathcal{F}$ onto $\mathcal{G}$ is of the form

$$Jf(x) = g(x) + f(\phi(x)) \tag{10}$$

where $g : \mathcal{X} \to \mathbb{R}$ is a continuous, function and $\phi : \mathcal{X} \to \mathcal{Y}$ is a homeomorphism. The same holds if l.s.c. is replaced by continuous, or if the functions are restricted to be proper.

Let $\mathcal{G}$ (resp. $\mathcal{F}$) be the set of Lipschitz functions over a complete metric space $(\mathcal{X}, d)$ (resp. $(\mathcal{Y}, d')$). Then every (max,+)-isoφ J from $\mathcal{F}$ onto $\mathcal{G}$ is of the form

$$Jf(x) = g(x) + f(\phi(x)) \tag{11}$$

where $g : \mathcal{X} \to \mathbb{R}$ is a Lipschitz function and $\phi : \mathcal{X} \to \mathcal{Y}$ is a bi-Lipschitz homeomorphism, i.e. ϕ and ϕ^{-1} are both Lipschitz.

Second application: c-convex functions

A kernel $b : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is fully-reduced if, for all x, y, $b(x, \cdot)$ and $b(\cdot, y)$ are sup-irreducible and, for all $x_0, x_1, y_0, y_1, \lambda \in \mathbb{R}$, $b(\cdot, y_0) = b(\cdot, y_1) + \lambda \implies y_0 = y_1$ and $b(x_0, \cdot) = b(x_1, \cdot) + \lambda \implies x_0 = x_1$.

Let $\mathcal{X}, \mathcal{X}', \mathcal{Y}, \mathcal{Y}'$ be Hausdorff compact topological spaces, and $b : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, $c : \mathcal{X}' \times \mathcal{Y}' \to \mathbb{R}$ be two continuous functions such that the kernels are fully-reduced. Then TFAE:

1. there exists a (max,+)-isoφ $J : \mathrm{Rg}(B) \to \mathrm{Rg}(C)$;
2. the two kernels satisfy that there exists two homeomorphisms $\tau : \mathcal{X}' \to \mathcal{X}$ and $\sigma : \mathcal{Y}' \to \mathcal{Y}$, and two continuous functions $\psi : \mathcal{X}' \to \mathbb{R}$ and $\varphi : \mathcal{Y}' \to \mathbb{R}$ such that

$$c(x', y') = \psi(x') + b(\tau(x'), \sigma(y')) + \varphi(y'). \tag{12}$$

3. all (max,+)-isoφ $J : \mathrm{Rg}(B) \to \mathrm{Rg}(C)$ are of the form $Jf = \psi + f \circ \tau$ for some $\psi : \mathcal{X}' \to \mathbb{R}$ and a bijective $\tau : \mathcal{X}' \to \mathcal{X}$, and there exists one such J.

Proof idea: show that the $b(\cdot, y) + \lambda$ are the only sup-irreducible functions.

Csq: Values of dual problems don't depend on the anti-involution!

We have that $g \in \mathrm{Rg}(B)$ iff $g = \bar{B}\bar{B}^\circ g$. We can commute max and min, to obtain weak duality:

$$\inf_{x\in\mathcal{X}} f(x) + g(x) = \inf_{x\in\mathcal{X}} f(x) + \sup_{y\in\mathcal{Y}} b(x,y) - \bar{B}^\circ g(y) \geq \sup_{y\in\mathcal{Y}} -\bar{B}^\circ g(y) + \inf_{x\in\mathcal{X}} f(x) + b(x,y).$$

Lemma (Unique dual value)

Let $\mathcal{G}$ be a complete subspace of $\overline{\mathbb{R}}^{\mathcal{X}}$. Take $f, g \in \overline{\mathbb{R}}^{\mathcal{X}}$ with $g \in \mathcal{G}$ and assume that $\mathcal{G} = \mathrm{Rg}(B) = \mathrm{Rg}(C)$ for $b : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ and $c : \mathcal{X} \times \mathcal{Y}' \to \mathbb{R}$. Assume furthermore that every $(\max,+)$-isoφ $J : \mathrm{Rg}(\bar{B}^\circ) \to \mathrm{Rg}(\bar{C}^\circ)$ is of the form $Jf = \psi + f \circ \tau$. Consider the primal problem $\inf_{x\in\mathcal{X}} f(x) + g(x)$, then

$$v = \sup_{y\in\mathcal{Y}}[\inf_{z\in\mathcal{X}}[g(z) - b(z,y)] + \inf_{x\in\mathcal{X}}[f(x) + b(x,y)]] = \sup_{y'\in\mathcal{Y}'}[\inf_{z\in\mathcal{X}}[g(z) - c(z,y')] + \inf_{x\in\mathcal{X}} f(x) + c(x,y')], \tag{13}$$

in other words, the value v of the dual problem does not depend on the kernel generating $\mathcal{G}$.

Example on the board

Definition

A map $\delta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ over a set $\mathcal{X}$ is a *weak metric* if, for all $x, y, z \in \mathcal{X}$, $\delta(x, x) = 0$, $\delta(x, y) \geq 0$ and $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$, and if $\delta(x, y) = \delta(y, x) = 0$ implies that $x = y$.
A map $f : \mathcal{X} \to \mathbb{R}$ is *nonexpansive* w.r.t. δ, or 1-Lipschitz, if $f(x) \leq \delta(x, y) + f(y)$ holds for all $x, y \in \mathcal{X}$. The set of 1-Lipschitz maps $f : \mathcal{X} \to \mathbb{R}$ is $\mathrm{Lip}_1(\mathcal{X}, \delta; \mathbb{R})$.

Showing, under some assumptions: *Busemann points are not of maximal Archimedean class*

Theorem

Let $\mathcal{G} = \mathrm{Lip}_1(\mathcal{X}, \delta; \mathbb{R})$ $\mathcal{F} = \mathrm{Lip}_1(\mathcal{Y}, \delta'; \mathbb{R})$).Assume either i) that the balls of $(\mathcal{X}, \delta_s)$ are compact or ii) that δ is symmetric and $(\mathcal{X}, \delta_s)$ is complete, and that the same is true for $(\mathcal{Y}, \delta')$. Then every (max,+)-isoφ J from $\mathcal{F}$ onto $\mathcal{G}$ is of the form

$$Jf(x) = g(x) + f(\phi(x)) \tag{14}$$

whith nonexpansive $g : \mathcal{X} \to \mathbb{R}$ and $\phi : \mathcal{X} \to \mathcal{Y}$ s.t. $g(x) - g(x') + \delta'(\phi(x), \phi(x')) = \delta(x, x')$ for all $x, x' \in \mathcal{X}$. If either δ or δ' is a metric, then g is constant and $\delta'(\phi(x), \phi(x')) = \delta(x, x')$.

Order isoφ of l.s.c. functions

Theorem

Let $\mathcal{G}$ (resp. $\mathcal{F}$) be the space of l.s.c. functions over a Hausdorff topological space $\mathcal{X}$ (resp. $\mathcal{Y}$). Then every max-isoφ J from $\mathcal{F}$ onto $\mathcal{G}$ is of the form

$$Jf(x) = g(x, f(\phi(x))) \tag{15}$$

where ϕ is a homeomorphism and $g : \mathcal{X} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is jointly l.s.c., and $g(x, \cdot)$ bijective and increasing for all $x \in \mathcal{X}$ with inverse $g^1(x, \cdot)$ such that $g^1(\cdot, \cdot)$ is also jointly l.s.c. The same holds if l.s.c. is replaced by continuous.

More generally, if $\mathcal{G}$ (resp. $\mathcal{F}$) is a pointwise-dense subset of the l.s.c. functions over a Hausdorff topological space $\mathcal{X}$ (resp. $\mathcal{Y}$) then (15) holds necessarily, with ϕ and g as above.

Actually for the structure (15) to hold, it suffices that $\{\delta_x^\top\}_{x\in\mathcal{X}} \subset \mathcal{G}$ and $\{\delta_y^\top\}_{y\in\mathcal{Y}} \subset \mathcal{F}$.

Theorem

Let $\mathcal{G}$ be the space of proper convex l.s.c. functions over a locally convex Hausdorff topological $\mathcal{X}$ of dimension larger than two, then the max-isoφ over $\mathcal{G}$ are affine, i.e. there exists a weakly-weakly continuous linear map $A : \mathcal{X} \to \mathcal{X}$ invertible with weakly-weakly continuous inverse, $c \in \mathcal{X}$, $b \in \mathcal{X}^*$, $d, \delta \in \mathbb{R}$ with $d > 0$ such that, for all $f \in \mathcal{G}$, we have

$$Jf(x) = \langle b, x \rangle + \delta + d \cdot f(Ax + c). \tag{16}$$

Theorem

Let $\mathcal{G}$ be the space of proper convex l.s.c. functions over a locally convex Hausdorff topological $\mathcal{X}$ of dimension larger than two, then the max-isoφ over $\mathcal{G}$ are affine, i.e. there exists a weakly-weakly continuous linear map $A : \mathcal{X} \to \mathcal{X}$ invertible with weakly-weakly continuous inverse, $c \in \mathcal{X}$, $b \in \mathcal{X}^*$, $d, \delta \in \mathbb{R}$ with $d > 0$ such that, for all $f \in \mathcal{G}$, we have

$$Jf(x) = \langle b, x \rangle + \delta + d \cdot f(Ax + c). \tag{16}$$

This is a consequence of

Proposition

The sup-irreducible points of the space of proper convex l.s.c. functions over a locally convex Hausdorff space $\mathcal{X}$ are the continuous affine maps $\langle p, \cdot \rangle + \lambda$ with $p \in \mathcal{X}^*$ and $\lambda \in \mathbb{R}$.

and of a **fundamental result of affine geometry**:

In dimensions larger than 2, transformations preserving straight lines are affine.

Corollary

Let $\mathcal{G}$ be the space of proper convex l.s.c. functions over a reflexive Banach space $\mathcal{X}$ of dimension larger than two, assumed to be linearly isomorphic to its dual $\mathcal{X}^*$. Then the anti-involutions over $\mathcal{G}$ are of the form

$$Tf(x) = \langle Kc, x \rangle + \delta + f^*(K(Ax + c)). \tag{17}$$

with $A \in \mathcal{L}(\mathcal{X})$ invertible, $c \in \mathcal{X}$, $\delta \in \mathbb{R}$, $K^{-1}A^{-\top}KA = \mathrm{Id}_{\mathcal{X}}$ and $(K - A^\top KA^{-1})c = 0$ where $K : \mathcal{X} \to \mathcal{X}^*$ is the duality operator.

This a generalization of [Artstein-Avidan and Milman, 2009] for which $\mathcal{X} = \mathbb{R}^d$, and of [Iusem et al., 2015] for which $\mathcal{X}$ is a Banach space.

Conclusion

For many sets, every (max,+)-isomorphism is of the form:

$$Jf(x) = g(x) + f(\phi(x))$$

Set $\mathcal{X}$	Function space $\mathcal{G}$	Translation g	Reparametrization ϕ
Hausdorff topological space	l.s.c. functions	continuous	homeomorphism
complete metric space	1-Lipschitz functions	constant	isometry
locally convex Hausdorff topological	l.s.c. convex functions	continuous affine	continuous affine

- For many other sets, orders isomorphisms are of the form $Jf(x) = g(x, f(\phi(x)))$.
- We encompass, simplify and extend a few previous works.
- When characterizing order isomorphisms, sup/inf-irreducible elements are nice invariants to focus on.

Conclusion

For many sets, every (max,+)-isomorphism is of the form:

$$Jf(x) = g(x) + f(\phi(x))$$

Set $\mathcal{X}$	Function space $\mathcal{G}$	Translation g	Reparametrization ϕ
Hausdorff topological space	l.s.c.	continuous	homeomorphism
metric space			try
topological	convex functions	continuous affine	continuous affine

Thank you for your attention!

https://arxiv.org/abs/2404.06857 **with *Stéphane Gaubert*. Comments much appreciated :)**

- For many other sets, orders isomorphisms are of the form $Jf(x) = g(x, f(\phi(x)))$.
- We encompass, simplify and extend a few previous works.
- When characterizing order isomorphisms, sup/inf-irreducible elements are nice invariants to focus on.

References I

Akian, M., Gaubert, S., and Kolokoltsov, V. N. (2005).
Set coverings and invertibility of functional Galois connections.
In Litvinov, G. L. and Maslov, V. P., editors, *Idempotent Mathematics and Mathematical Physics*, Contemporary Mathematics, pages 19–51. American Mathematical Society.

Artstein-Avidan, S. and Milman, V. (2009).
The concept of duality in convex analysis, and the characterization of the Legendre transform.
Annals of Mathematics, 169(2):661–674.

Artstein-Avidan, S., Sadovsky, S., and Wyczesany, K. (2023).
A zoo of dualities.
The Journal of Geometric Analysis, 33(8).

Berndtsson, B., Cordero-Erausquin, D., Klartag, B., and Rubinstein, Y. A. (2020).
Complex legendre duality.
American Journal of Mathematics, 142(1):323–339.

References II

Iusem, A. N., Reem, D., and Svaiter, B. F. (2015).
Order preserving and order reversing operators on the class of convex functions in Banach spaces.
Journal of Functional Analysis, 268(1):73–92.

Lempert, L. (2017).
On complex legendre duality.
The Journal of Geometric Analysis, 30(3):2581–2592.

Leung, D. H. and Tang, W.-K. (2016).
Nonlinear order isomorphisms on function spaces.
Dissertationes Mathematicae, page 1–74.

Singer, I. (1984).
Conjugation operators.
In *Lecture Notes in Economics and Mathematical Systems*, pages 80–97. Springer Berlin Heidelberg.

References III

Singer, I. (1997).
Abstract Convex Analysis.
Wiley-Interscience and Canadian Mathematics Series of Monographs and Texts. Wiley-Interscience, 1 edition.

Weaver, N. (1994).
Lattices of lipschitz functions.
Pacific Journal of Mathematics, 164(1):179–193.