State-constrained Linear-Quadratic Optimal Control in light of the LQ reproducing kernel

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https://pcaubin.github.io/

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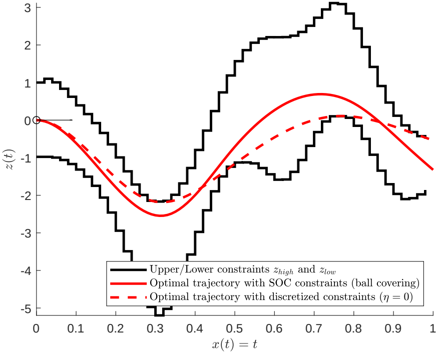
What are state constraints?

Estimation

Control



Side information

$\hookrightarrow$ compensates small number of samples or excessive noise

Physical constraints

$\hookrightarrow$ provides feasible trajectories in path-planning

Ubiquitous and both handled as a constrained optimization problem

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Linearly-constrained Linear Quadratic Regulator from the viewpoint of kernel methods, June 2020, `https://arxiv.org/abs/2011.02196`

Optimal control as optimization over vector spaces

Optimization over $\mathbb{R}^N$

$$\min_{x \in \mathbb{R}^N} \quad \mathcal{L}(x)$$

$$\text{s.t.}$$

$$g_i(x) = 0 \leftrightarrow \nu_i \in \mathbb{R},$$

$$h_j(x) \leq 0 \leftrightarrow \mu_j \in \mathbb{R}_+$$

Optimal control as optimization over vector spaces

Optimization over Hilbert spaces

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e.g. $\mathcal{F}$ a Sobolev space $H^1(\mathbb{R}^N, \mathbb{R})$

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Nonlinear Optimal Control

$$\min_{\substack{x(\cdot) \in \mathcal{C}^0, u(\cdot) \in L^p \\ \text{s.t.}}} \mathcal{L}(x(\cdot), u(\cdot))$$

$$x'(t) \overset{\text{a.e.}}{=} f(x, u) \leftrightarrow p(t) \in \mathbb{R}^N,$$
$$h_j(x(t)) \leq 0, \forall t \quad \leftrightarrow \psi_j(\cdot) \in \text{BV}$$

Non-reflexive Banach spaces...

Optimal control as optimization over vector spaces

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Linear Optimal Control

$$\min_{x(\cdot) \in \mathcal{C}^0, u(\cdot) \in L^2} \quad \mathcal{L}(x(\cdot), u(\cdot))$$

s.t.

$$x'(t) \overset{\text{a.e.}}{=} Ax + Bu \leftrightarrow p(t) \in \mathbb{R}^N,$$

$$Cx(t) \leq d(t), \forall t \quad \leftrightarrow \psi_j(\cdot) \in \mathrm{BV}$$

(Reflexive) Hilbert spaces?

Optimal control as optimization over vector spaces

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(Reflexive) Hilbert spaces?

How do you compute on an infinite dimensional space?

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(Reflexive) Hilbert spaces?

How do you compute on an infinite dimensional space?

How do you compute with BV dual vectors?

Optimal control as optimization over vector spaces

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(Reflexive) Hilbert spaces?

How do you compute on an infinite dimensional space?

What if your Hilbert space was not any Hilbert space but a RKHS?

Reproducing kernel Hilbert spaces (RKHS) at a glance (1)

A RKHS $(\mathcal{F}_k, \langle \cdot, \cdot \rangle_{\mathcal{F}_k})$ is a Hilbert space of real-valued[1] functions over a set $\mathcal{T}$ if one of the following equivalent conditions is satisfied [Aronszajn, 1950]

$\exists\, k : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ s.t. $k_t(\cdot) = k(\cdot, t) \in \mathcal{F}_k$ and $f(t) = \langle f(\cdot), k_t(\cdot) \rangle_{\mathcal{F}_k}$

the topology of $(\mathcal{F}_k, \langle \cdot, \cdot \rangle_{\mathcal{F}_k})$ is stronger than pointwise convergence i.e. $\delta_t : f \mapsto f(t)$ is continuous for all x for $f \in \mathcal{F}_k$.

$$|f(t) - f_n(t)| = |\langle f - f_n, k_t \rangle_k| \leq \|f - f_n\|_k \|k_t\|_k = \|f - f_n\|_k \sqrt{k(t,t)}$$

[1] There is a natural extension to vector-valued RKHSs (more on this later).

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k is s.t. $\exists\, \Phi_k : \mathcal{T} \to \mathcal{F}_k$ s.t. $k(t,s) \; = \; \langle \Phi_k(t), \Phi_k(s) \rangle_{\mathcal{F}_k}$, $\Phi_k(t) = k_t(\cdot)$

k is s.t. $\mathbf{G} = [k(t_i, t_j)]_{i,j=1}^n \succcurlyeq 0$ and $\mathcal{F}_k := \overline{\operatorname{span}(\{k_t(\cdot)\}_{t \in \mathcal{T}})}$, i.e. the completion for the pre-scalar product $\langle k_t(\cdot), k_s(\cdot) \rangle_{k,0} = k(t,s)$

[1] There is a natural extension to vector-valued RKHSs (more on this later).

Reproducing kernel Hilbert spaces (RKHS) at a glance (2)

- There is a one-to-one correspondence between kernels k and RKHSs $(\mathcal{F}_k, \langle \cdot, \cdot \rangle_{\mathcal{F}_k})$. Changing $\mathcal{T}$ or $\langle \cdot, \cdot \rangle_{\mathcal{F}_k}$ changes the kernel k.

- For $\mathcal{T} \subset \mathbb{R}^d$, Sobolev spaces $\mathcal{H}^s(\mathcal{T})$ satisfying $s > d/2$ are RKHSs. For $\mathcal{T} = \mathbb{R}^d$ their (Matérn) kernels are well known. Classical kernels include

$$k_{\text{Gauss}}(t,s) = \exp\left(-\|t-s\|^2_{\mathbb{R}^d}/(2\sigma^2)\right) \quad k_{\text{lin}}(t,s) = \langle t, s \rangle_{\mathbb{R}^d}$$

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Identifying k and $\mathcal{F}_k$ is hard! Example for Sobolev spaces, $d = 1$, $\mathcal{T} = \mathbb{R}_+$

- H^1 with $\langle f, g \rangle_{H^1} = \frac{2}{\pi} \int_0^\infty fg + f'g' dt$

$$k(t,s) = \frac{\pi}{4} \left(\exp(-|t-s|) + \exp(-t-s)\right).$$

- H^1_0 $(f(0) = 0)$ with $\langle \cdot, \cdot \rangle_{H^1}$

$$k(t,s) = \frac{\pi}{4} \left(\exp(-|t-s|) - \exp(-t-s)\right).$$

- H^1_0 with $\langle f, g \rangle_{H^1_0} = \int_0^\infty f'g' dt$ has for kernel $k(t,s) = \min(t,s)$.

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Two essential tools for computations

Representer Theorem (e.g. [Schölkopf et al., 2001])

Let $L : (\mathcal{T} \times \mathbb{R} \times \mathbb{R})^N \to \mathbb{R} \cup \{\infty\}$, strictly increasing $\Omega : \mathbb{R}_+ \to \mathbb{R}$, and

$$\bar{f} \in \arg\min_{f \in \mathcal{F}_k} L\left(\left(t_n, y_n, f(t_n)\right)_{n\in[N]}\right) + \Omega\left(\|f\|_k\right)$$

Then $\exists\, (a_n)_{n\in[N]} \in \mathbb{R}^N$ s.t. $\bar{f}(\cdot) = \sum_{n\in[N]} a_n k(t_n, \cdot)$

$\hookrightarrow$ Optimal solutions lie in a finite dimensional subspace of $\mathcal{F}_k$.

Finite number of evaluations $\implies$ finite number of coefficients

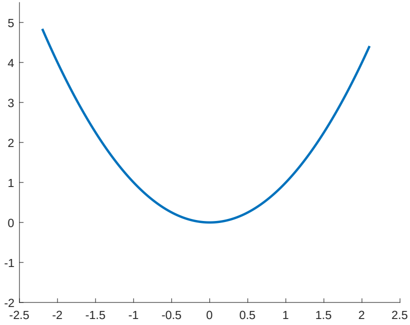
Kernel trick

$$\langle \sum_{n\in[N]} a_n k(t_n, \cdot), \sum_{m\in[M]} b_m k(s_m, \cdot)\rangle_k = \sum_{n\in[N]} \sum_{m\in[N']} a_n b_m k(t_n, s_m)$$

$\hookrightarrow$ On this finite dimensional subspace, no need to know $(\mathcal{F}_k, \langle \cdot, \cdot \rangle_{\mathcal{F}_k})$.

Example: 1D kernel ridge regression (KRR)

Estimating a function



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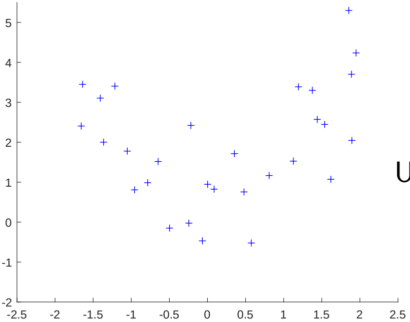

$$\min_{f\in\mathcal{F}_k} \frac{1}{N}\sum_{n=1}^{N} |y_n - f(t_n)|^2 + \lambda \|f\|^2_{\mathcal{F}_k}$$

Applying the representer theorem

Unconstrained KRR $\bar{f} = \sum_{n=1}^{N} \alpha_n k_{t_n}$,

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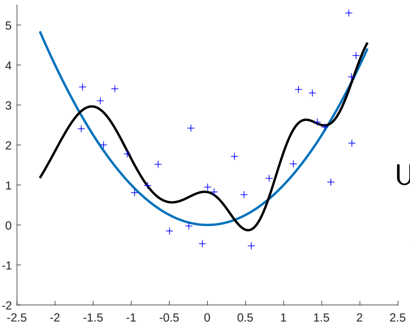

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Applying the representer theorem

Unconstrained KRR $\bar{f} = \sum_{n=1}^{N} \alpha_n k_{t_n}$,

$$\boldsymbol{\alpha} = (\boldsymbol{G} + N\lambda \cdot \mathrm{Id})^{-1} \boldsymbol{y}$$

with Gram matrix $\boldsymbol{G} = [k(t_i, t_j)]_{i,j \leq N}$

$$\sum_{n=1}^{N} \left| y_n - \sum_{i=1}^{N} \alpha_i k(t_i, t_n) \right|^2 + N\lambda \left\| \sum_{n=1}^{N} \alpha_n k_{t_n}(\cdot) \right\|_{\mathcal{F}_k}^2 = \|\boldsymbol{y} - \boldsymbol{G}\boldsymbol{\alpha}\|_{\mathbb{R}^N}^2 + N\lambda \boldsymbol{\alpha}^\top \boldsymbol{G}\boldsymbol{\alpha}$$

Convex and differentiable. Optimality: $0 = \boldsymbol{G}(-\boldsymbol{y} + \boldsymbol{G}\boldsymbol{\alpha} + N\lambda\boldsymbol{\alpha})$

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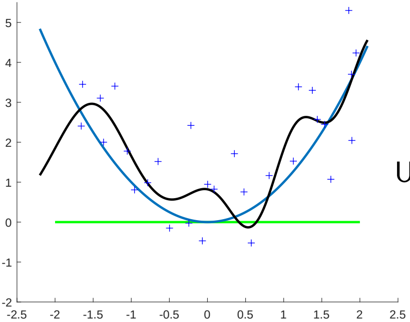

$$\min_{f \in \mathcal{F}_k} \frac{1}{N} \sum_{n=1}^{N} |y_n - f(t_n)|^2 + \lambda \|f\|_{\mathcal{F}_k}^2$$

s.t. $0 \leq f(t), \forall t \in [-2, 2]$

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here is not nonnegative around 0.5!

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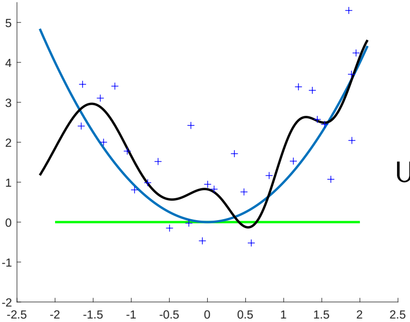

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Infinite number of evaluations $\Rightarrow$ no representer theorem!

How to modify the problem to ensure constraint satisfaction?

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Linearly-constrained Linear Quadratic Regulator (LQR)

Problem of time-varying linear quadratic optimal control with finite horizon and affine inequality state constraints

$$\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & g(x(T)) + \int_0^T [x(t)^\top Q(t)x(t) + u(t)^\top R(t)u(t)]\mathrm{d}t \\
\text{s.t.} \quad & \\
& x(0) = x_0, \\
& x'(t) = A(t)x(t) + B(t)u(t), \text{ a.e. in } [0, T], \\
& c_i(t)^\top x(t) \le d_i(t), \forall\, t \in [0, T], \forall\, i \in [\![1, P]\!],
\end{aligned}$$

with state $x(t) \in \mathbb{R}^N$, control $u(t) \in \mathbb{R}^M$, $A(\cdot) \in L^1(0, T)$, $B(\cdot) \in L^2(0, T)$, $Q(t) \succcurlyeq 0$ and $R(t) \succcurlyeq r\mathrm{Id}_M$ $(r > 0)$

Linearly-constrained Linear Quadratic Regulator (LQR)

Problem of time-varying linear quadratic optimal control with finite horizon and affine inequality state constraints with $Q \equiv 0$ and $R \equiv \mathrm{Id}_M$

$$\min_{x(\cdot),u(\cdot)} \quad g(x(T)) + \int_0^T \|u(t)\|_{\mathbb{R}^M}^2 \,\mathrm{d}t$$

$$\text{s.t.}$$

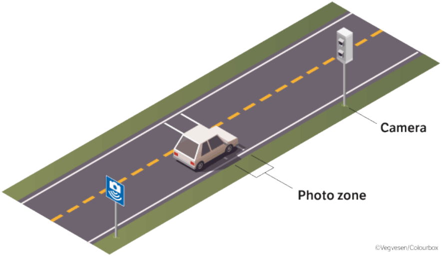
$$\begin{aligned} x(0) &= x_0, \\ x'(t) &= A(t)x(t) + B(t)u(t), \text{ a.e. in } [0, T], \\ c_i(t)^\top x(t) &\le d_i(t), \ \forall\, t \in [0, T], \forall\, i \in \{1, \dots, P\}, \end{aligned}$$

with state $x(t) \in \mathbb{R}^N$, control $u(t) \in \mathbb{R}^M$, $A(\cdot) \in L^1(0, T)$, $B(\cdot) \in L^2(0, T)$, $x(\cdot) : [0, T] \to \mathbb{R}^N$ absolutely continuous and $u(\cdot) \in L^2(0, T)$.

Why are state constraints difficult to study?

- **Theoretical obstacle**: Pontryagine's Maximum Principle involves not only an adjoint vector $p(t)$ but also measures/BV functions $\psi(t)$ supported at times where the constraints are saturated. You cannot just backpropagate the Hamiltonian system from the transversality condition.

- **Numerical obstacle**: Time discretization of constraints may fail e.g.



Speed cameras in traffic control

In between two cameras, drivers always break the speed limit.

Objective: Turn the state-constrained LQR into "KRR"

We have a vector space $\mathcal{S}$ of controlled trajectories $x(\cdot) : [0, T] \to \mathbb{R}^N$

$$\mathcal{S} := \{x(\cdot) \,|\, \exists\, u(\cdot) \in L^2(0, T) \text{ s.t. } x'(t) = A(t)x(t) + B(t)u(t) \text{ a.e. } \}$$

The space of controlled trajectories $\mathcal{S}$ depends on $[0, T]$, $A(\cdot)$, $B(\cdot)$.

LQR (Linear Quadratic Regulator)

$$\min_{\substack{x(\cdot)\in\mathcal{S}\\ u(\cdot)\in L^2(0,T)}} g(x(T)) + \|u(\cdot)\|^2_{L^2(0,T)}$$

$x(0) = x_0$
$c_i(t)^\top x(t) \le d_i(t), t \in [0, T], i \le P$

"KRR" (Kernel Ridge Regression)

$$\min_{x(\cdot)\in\mathcal{S}} g(x(T)) + \lambda \|x(\cdot)\|^2_{\mathcal{S}}$$

$x(0) = x_0$
$c_i(t)^\top x(t) \le d_i(t), t \in [0, T], i \le P$

Is $\mathcal{S}$ a RKHS? For which inner product?

Vector-valued reproducing kernel Hilbert space (vRKHS)

Definition (vRKHS)

Let $\mathcal{T}$ be a non-empty set. A Hilbert space $(\mathcal{F}_K, \langle \cdot, \cdot \rangle_K)$ of $\mathbb{R}^N$-vector-valued functions defined on $\mathcal{T}$ is a vRKHS if there exists a matrix-valued kernel $K : \mathcal{T} \times \mathcal{T} \to \mathbb{R}^{N \times N}$ such that the reproducing property holds:

$$K(\cdot, t)p \in \mathcal{F}_K, \quad p^\top f(t) = \langle f, K(\cdot, t)p \rangle_K, \qquad \text{for } t \in \mathcal{T},\ p \in \mathbb{R}^N, f \in \mathcal{F}_K$$

Necessarily, K has a Hermitian symmetry: $K(s,t) = K(t,s)^\top$

There is also a one-to-one correspondence between K and $(\mathcal{F}_K, \langle \cdot, \cdot \rangle_K)$ [Micheli and Glaunès, 2014], so changing $\mathcal{T}$ or $\langle \cdot, \cdot \rangle_K$ changes K.

Representer theorem in vRKHSs

Theorem (Representer theorem with constraints, Aubin 2020)

Let $(\mathcal{F}_K, \langle\cdot,\cdot\rangle_K)$ be a vRKHS defined on a set $\mathcal{T}$. For a "loss" $L : \mathbb{R}^{N_0} \to \mathbb{R} \cup \{+\infty\}$, strictly increasing "regularizer" $\Omega : \mathbb{R}_+ \to \mathbb{R}$, and constraints $d_i : \mathbb{R}^{N_i} \to \mathbb{R}$, consider the optimization problem

$$\bar{f} \in \underset{f\in\mathcal{F}_K}{\arg\min} \quad L\left(c_{0,1}^\top f(t_{0,1}), \ldots, c_{0,N_0}^\top f(t_{0,N_0})\right) + \Omega\left(\|f\|_K\right)$$

$$\text{s.t.}$$

$$\lambda_i \|f\|_K \le d_i(c_{i,1}^\top f(t_{i,1}), \ldots, c_{i,N_i}^\top f(t_{i,N_i})), \, \forall\, i \in [\![1, P]\!].$$

Then there exists $\{p_{i,m}\}_{m\in[\![1,N_i]\!]} \subset \mathbb{R}^N$ and $\alpha_{i,m} \in \mathbb{R}$ such that

$$\bar{f} = \sum_{i=0}^{P} \sum_{m=1}^{N_i} K(\cdot, t_{i,m}) p_{i,m} \text{ with } p_{i,m} = \alpha_{i,m} c_{i,m}.$$

Application to linear control systems with quadratic cost

$\mathcal{S} := \{x(\cdot) \in W^{1,1} \mid \exists\, u(\cdot) \in L^2(0,T) \text{ s.t. } x'(t) = A(t)x(t)+B(t)u(t) \text{ a.e. }\}$

Given $x(\cdot) \in \mathcal{S}$, for the pseudoinverse $B(t)^{\ominus}$ of $B(t)$, set

$$u(t) := B(t)^{\ominus}[x'(t) - A(t)x(t)] \;\text{ a.e. in } [0,T].$$

$$\langle x_1(\cdot), x_2(\cdot)\rangle_K := x_1(0)^\top x_2(0) + \int_0^T u_1(t)^\top u_2(t)\mathrm{d}t$$

Lemma

$(\mathcal{S}, \langle\cdot,\cdot\rangle_K)$ is a vRKHS with uniformly continuous $K(\cdot,\cdot)$.

$\|\cdot\|_K$ is a Sobolev-like norm split into two semi-norms

$$\|x(\cdot)\|_K^2 = \underbrace{\|x(0)\|^2}_{\|x(\cdot)\|_{K_0}^2} + \underbrace{\int_0^T \|B(t)^{\ominus}(x'(t) - A(t)x(t))\|^2\mathrm{d}t}_{\|x(\cdot)\|_{K_1}^2}.$$

Splitting $\mathcal{S}$ into subspaces and identifying their kernels

$$\mathcal{S}_0 := \{x(\cdot) \,|\, x'(t) = A(t)x(t), \text{ a.e. in } [0, T]\} \qquad \|x(\cdot)\|_{K_0}^2 = \|x(0)\|^2$$

$$\mathcal{S}_u := \{x(\cdot) \,|\, x(\cdot) \in \mathcal{S} \text{ and } x(0) = 0\} \qquad \|x(\cdot)\|_{K_1}^2 = \|u(\cdot)\|_{L^2(0,T)}^2.$$

As $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_u$, $K = K_0 + K_1$.

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$$\mathcal{S}_u := \{x(\cdot) \mid x(\cdot) \in \mathcal{S} \text{ and } x(0) = 0\} \qquad \|x(\cdot)\|_{K_1}^2 = \|u(\cdot)\|_{L^2(0,T)}^2.$$

As $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_u$, $K = K_0 + K_1$. Since $\dim(\mathcal{S}_0) = N$, for $\Phi_A(t,s) \in \mathbb{R}^{N,N}$ the state-transition matrix $s \to t$ of $x'(\tau) = A(\tau)x(\tau)$

$$K_0(s,t) = \Phi_A(s,0)\Phi_A(t,0)^\top$$

Splitting $\mathcal{S}$ into subspaces and identifying their kernels

$\mathcal{S}_0 := \{x(\cdot) \,|\, x'(t) = A(t)x(t), \text{ a.e. in } [0, T]\}$ $\qquad \|x(\cdot)\|^2_{K_0} = \|x(0)\|^2$

$\mathcal{S}_u := \{x(\cdot) \,|\, x(\cdot) \in \mathcal{S} \text{ and } x(0) = 0\}$ $\qquad \|x(\cdot)\|^2_{K_1} = \|u(\cdot)\|^2_{L^2(0,T)}$.

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$$K_0(s,t) = \Phi_A(s,0)\Phi_A(t,0)^\top$$

Identify K_1 using only the reproducing property and that for $x(\cdot) \in \mathcal{S}$,

$$x(t) = \Phi_A(t,0)x(0) + \int_0^t \Phi_A(t,\tau)B(\tau)u(\tau)\mathrm{d}\tau. \qquad (1)$$

For fixed t, define a control matrix $U_t(s) := \begin{cases} B(s)^\top \Phi_A(t,s)^\top & \forall s \leq t, \\ 0 & \forall s > t. \end{cases}$

$$\partial_1 K_1(s,t) = A(s)K_1(s,t) + B(s)U_t(s) \text{ a.e. in } [0,T] \text{ with } K_1(0,t) = 0.$$

$$K_1(s,t) = \int_0^{\min(s,t)} \Phi_A(s,\tau)B(\tau)B(\tau)^\top \Phi_A(t,\tau)^\top \mathrm{d}\tau.$$

Examples: controllability Gramian/transversality condition

Steer a point from $(0,0)$ to (T, x_T), with e.g. $g(x(T)) = \|x_T - x(T)\|_N^2$

Exact planning ($x(T) = x_T$)

$$\min_{\substack{x(\cdot)\in\mathcal{S}\\ x(0)=0}} \chi_{x_T}(x(T)) + \frac{1}{2}\|u(\cdot)\|^2_{L^2(0,T)}$$

Relaxed planning ($g \in \mathcal{C}^1$ convex)

$$\min_{\substack{x(\cdot)\in\mathcal{S}\\ x(0)=0}} g(x(T)) + \frac{1}{2}\|u(\cdot)\|^2_{L^2(0,T)}$$

As, $x(0) = 0$, applying the representer theorem: $\exists\, p_T$, $\bar{x}(\cdot) = K_1(\cdot, T)p_T$

Controllability Gramian

$$K_1(T,T) = \int_0^T \Phi_A(T,\tau)B(\tau)B(\tau)^\top \Phi_A(T,\tau)^\top \mathrm{d}\tau$$

$\bar{x}(T) = x_T \Leftrightarrow x_T \in \operatorname{Im}(K_1(T,T))$

Transversality Condition

$$0 = \nabla\left(p \mapsto g(K_1(T,T)p) + \frac{1}{2}p^\top K_1(T,T)p\right)(p_T)$$
$$= K_1(T,T)(\nabla g(K_1(T,T)p_T) + p_T).$$

Take $p_T = -\nabla g(\bar{x}(T))$

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Dealing with an infinite number of constraints

No representer theorem for: $c(t)^\top x(t) \leq d, \forall t \in [0, T]$

Discretize on $\{t_m\}_{m\in[M]} \subset [0, T]$?

$$c(t_m)^\top x(t_m) \leq d, \forall m \in [\![1, M]\!]$$

No guarantees!

Dealing with an infinite number of constraints

No representer theorem for: $c(t)^\top x(t) \leq d, \forall t \in [0, T]$

Discretize on $\{t_m\}_{m\in[M]} \subset [0, T]$?

$$\eta_m \|x(\cdot)\|_K + c(t_m)^\top x(t_m) \leq d, \forall m \in [\![1, M]\!]$$

Second-Order Cone (SOC) constraints: $\{f \mid \|Af + b\|_K \leq c^\top f + d\}$

SOC comes from adding a buffer, $\eta_m > 0$, to a discretization, $\{t_m\}_{m\in[M]}$

LP$\subset$ QP $\subset$ SOCP $\subset$ SDP

Dealing with an infinite number of constraints

No representer theorem for: $c(t)^\top x(t) \leq d, \forall t \in [0, T]$

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Second-Order Cone constraints: $\{f \mid \|Af + b\|_K \leq c^\top f + d\}$

SOC comes from adding a buffer, $\eta_m > 0$, to a discretization, $\{t_m\}_{m\in[M]}$.

The choice $\eta_m \|x(\cdot)\|_K$ is related to continuity moduli: How to choose η_m?

Deriving SOC constraints through continuity moduli

Take $\delta \geq 0$ and t s.t. $|t - t_m| \leq \delta$

$$|c(t)^\top x(t) - c(t_m)^\top x(t_m)| = |\langle x(\cdot), K(\cdot,t)c(t) - K(\cdot,t_m)c(t_m)\rangle_K|$$
$$\leq \|x(\cdot)\|_K \underbrace{\sup_{\{t \,|\, |t-t_m|\leq\delta\}} \|K(\cdot,t)c(t) - K(\cdot,t_m)c(t_m)\|_K}_{\eta_m(\delta)}$$

$$\omega_m(x,\delta) := \sup_{\{t \,|\, |t-t_m|\leq\delta\}} |c(t)^\top x(t) - c(t_m)^\top x(t_m)| \leq \eta_m(\delta)\|x(\cdot)\|_K$$

For a covering $[0, T] = \bigcup_{m\in[M]}[t_m - \delta_m, t_m + \delta_m]$

$$“c(t)^\top x(t) \leq d, \forall t \in [0,T]“ \Leftrightarrow “c(t_m)^\top x(t_m) + \omega_m(x,\delta) \leq d, \forall m \in [M]“$$

Deriving SOC constraints through continuity moduli

Take $\delta \geq 0$ and t s.t. $|t - t_m| \leq \delta$

$$|c(t)^\top x(t) - c(t_m)^\top x(t_m)| = |\langle x(\cdot), K(\cdot,t)c(t) - K(\cdot,t_m)c(t_m)\rangle_K|$$
$$\leq \|x(\cdot)\|_K \underbrace{\sup_{\{t \,|\, |t-t_m|\leq\delta\}} \|K(\cdot,t)c(t) - K(\cdot,t_m)c(t_m)\|_K}_{\eta_m(\delta)}$$

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For a covering $[0,T] = \bigcup_{m\in[M]}[t_m - \delta_m, t_m + \delta_m]$

$$\text{“}c(t)^\top x(t) \leq d \quad, \forall t \in [0,T]\text{“} \Leftarrow \text{“}c(t_m)^\top x(t_m) + \eta_m\|x(\cdot)\| \leq d \quad, \forall m \in [M]\text{“}$$

$$\|K(\cdot,t)c(t) - K(\cdot,t_m)c(t_m)\|_K^2 := c(t)^\top K(t,t)c(t) + c(t_m)^\top K(t_m,t_m)c(t_m) - 2c(t_m)^\top K(t_m,t)c(t)$$

Since the kernel is smooth, for $c(\cdot) \in \mathcal{C}^0$, $\delta \to 0$ gives $\eta_m(\delta) \to 0$.

Deriving SOC constraints through continuity moduli

Take $\delta \geq 0$ and t s.t. $|t - t_m| \leq \delta$

$$|c(t)^\top x(t) - c(t_m)^\top x(t_m)| = |\langle x(\cdot), K(\cdot, t)c(t) - K(\cdot, t_m)c(t_m)\rangle_K|$$
$$\leq \|x(\cdot)\|_K \underbrace{\sup_{\{t \,|\, |t-t_m|\leq\delta\}} \|K(\cdot, t)c(t) - K(\cdot, t_m)c(t_m)\|_K}_{\eta_m(\delta)}$$

$$\omega_m(x, \delta) := \sup_{\{t \,|\, |t-t_m|\leq\delta\}} |c(t)^\top x(t) - c(t_m)^\top x(t_m)| \leq \eta_m(\delta)\|x(\cdot)\|_K$$

For a covering $[0, T] \subset \bigcup_{m\in[M]} [t_m - \delta_m, t_m + \delta_m]$

$$\text{“}c(t)^\top x(t) \leq d(t), \forall t \in [0, T]\text{“} \Leftarrow \text{“}c(t_m)^\top x(t_m) + \eta_m \|x(\cdot)\| \leq d_m, \forall m \in [M]\text{“}$$

with $d_m := \inf_{t\in[t_m-\delta_m, t_m+\delta_m]} d(t)$.

From affine state constraints to SOC constraints

Take (t_m, δ_m) such that $[0, T] \subset \cup_{m \in [\![1, N_P]\!]} [t_m - \delta_m, t_m + \delta_m]$, define

$$\eta_i(\delta_m, t_m) := \sup_{t \in [t_m - \delta_m, t_m + \delta_m] \cap [0,T]} \|K(\cdot, t_m) c_i(t_m) - K(\cdot, t) c_i(t)\|_K,$$

$$d_i(\delta_m, t_m) := \inf_{t \in [t_m - \delta_m, t_m + \delta_m] \cap [0,T]} d_i(t).$$

We have strengthened SOC constraints that enable a representer theorem

$$\eta_i(\delta_m, t_m)\|x(\cdot)\|_K + c_i(t_m)^\top x(t_m) \le d_i(\delta_m, t_m), \; \forall\, m \in [\![1, N_P]\!], \forall\, i \in [\![1, P]\!]$$

$$\Downarrow$$

$$c_i(t)^\top x(t) \le d_i(t), \; \forall\, t \in [0, T], \forall\, i \in [\![1, P]\!]$$

Lemma (Uniform continuity of tightened constraints)

As $K(\cdot, \cdot)$ is UC, if $c_i(\cdot)$ and $d_i(\cdot)$ are $\mathcal{C}^0$-continuous, when $\delta \to 0^+$, $\eta_i(\cdot, t)$ converges to 0 and $d_i(\cdot, t)$ converges to $d_i(t)$, uniformly w.r.t. t.

Main theorem [Aubin-Frankowski, 2020]

(H-gen) $A(\cdot) \in L^1(0,T)$ and $B(\cdot) \in L^2(0,T)$, $c_i(\cdot)$ and $d_i(\cdot)$ are $\mathcal{C}^0$.

(H-sol) $c_i(0)x_0 < d_i(0)$ and there exists a trajectory $x^\epsilon(\cdot) \in \mathcal{S}$ satisfying strictly the affine constraints, as well as the initial condition.[2]

(H-obj) $g(\cdot)$ is convex and continuous.

Theorem (Existence and Approximation by SOC constraints)

Both the original problem and its strengthening have unique optimal solutions. For any $\rho > 0$, *there exists* $\overline{\delta} > 0$ *such that for all* $(\delta_m)_{m \in [\![1,N_0]\!]}$, *with* $[0,T] \subset \cup_{m \in [\![1,N_0]\!]}[t_m - \delta_m, t_m + \delta_m]$ *satisfying* $\overline{\delta} \geq \max_{m \in [\![1,N_0]\!]} \delta_m$,

$$\frac{1}{\gamma_K} \cdot \sup_{t \in [0,T]} \|\bar{x}_\eta(t) - \bar{x}(t)\| \leq \|\bar{x}_\eta(\cdot) - \bar{x}(\cdot)\|_K \leq \rho.$$

with $\gamma_K := \sup_{t \in [0,T],\, p \in \mathbb{B}_N} \sqrt{p^\top K(t,t)p}$.

[2] (H-sol) is implied for instance by an inward-pointing condition at the boundary.

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To solve a state-constrained LQR through kernel regression, we have to

- identify the kernel $K(s, t)$ related to $[0, T]$, A and B (and Q and R)
- strengthen the infinite affine constraints to finite SOC constraints
- apply a representer theorem to the SOC-tightened problem
- solve a finite dimensional SOCP over the covectors p_t

Two examples: a submarine and a pendulum

Numerical example: submarine in a cavern

Original control problem

$$\min_{z(\cdot)\in W^{2,1}, u(\cdot)\in L^2(\mathcal{T},\mathbb{R})} \quad \int_{\mathcal{T}} |u(t)|^2 \mathrm{d}t$$

$$\text{s.t.}$$

$$z(0) = 0, \quad \dot{z}(0) = 0,$$

$$\ddot{z}(t) = -\dot{z}(t) + u(t), \forall t \in \mathcal{T},$$

$$z(t) \in [z_{\text{low}}(t), z_{\text{up}}(t)], \forall\, t \in \mathcal{T}.$$

Numerical example: submarine in a cavern

Original control problem

$$\min_{z(\cdot)\in W^{2,1}, u(\cdot)\in L^2(\mathcal{T},\mathbb{R})} \int_{\mathcal{T}} |u(t)|^2 \mathrm{d}t$$

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$$z(t) \in [z_{\text{low}}(t), z_{\text{up}}(t)], \; \forall\, t \in \mathcal{T}.$$

Rewriting in standard form

$$\min_{x(\cdot)\in \mathcal{C}^0, u(\cdot)\in L^2} \int_{\mathcal{T}} |u(t)|^2 \mathrm{d}t$$

s.t.

$$x(0) = 0,$$

$$x'(t) \overset{\text{a.e.}}{=} Ax(t) + Bu(t),$$

$$x_1(t) \in [z_{\text{low}}(t), z_{\text{up}}(t)], \; \forall\, t \in \mathcal{T}$$

$$x = \begin{pmatrix} z \\ \dot{z} \end{pmatrix}, \; A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Numerical example: submarine in a cavern

RKHS regression

$$\min_{x(\cdot) \in \mathcal{S}_u} \quad \|x(\cdot)\|^2_{K_1}$$
$$\text{s.t.}$$
$$x_1(t) \in [z_{\text{low}}(t), z_{\text{up}}(t)], \ \forall\, t \in \mathcal{T}$$

Rewriting in standard form

$$\min_{x(\cdot)\in\mathcal{C}^0, u(\cdot)\in L^2} \quad \int_{\mathcal{T}} |u(t)|^2 \mathrm{d}t$$
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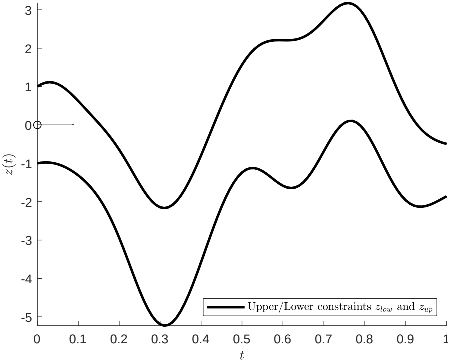
Numerical example: submarine in a cavern

RKHS regression

$$\min_{x(\cdot) \in \mathcal{S}_u} \quad \|x(\cdot)\|_{K_1}^2$$

s.t.

$$x_1(t) \in [z_{\text{low}}(t), z_{\text{up}}(t)], \; \forall\, t \in \mathcal{T}$$



$$\mathcal{S}_u := \{x(\cdot) \,|\, x(\cdot) \in \mathcal{S} \text{ and } x(0) = 0\} \quad \|x(\cdot)\|_{K_1}^2 = \|u(\cdot)\|_{L^2(0,T)}^2.$$

$$K_1(s,t) = \int_0^{\min(s,t)} e^{(s-\tau)A} BB^\top e^{(t-\tau)A^\top} \mathrm{d}\tau$$

Numerical example: submarine in a cavern

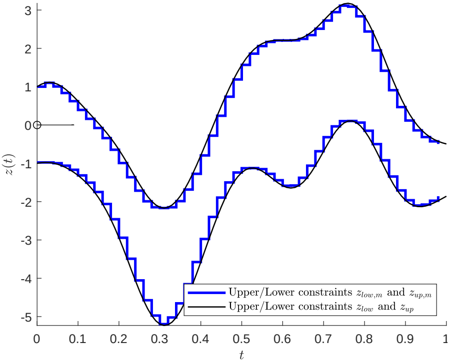
RKHS regression

$$\min_{x(\cdot) \in \mathcal{S}_u} \quad \|x(\cdot)\|_{K_1}^2$$

s.t.

$$x_1(t) \in [z_{\text{low},m}, z_{\text{up},m}],$$

$$\forall\, t \in [t_m - \delta_m, t_m + \delta_m],\ \forall m \in [M]$$



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Numerical example: submarine in a cavern

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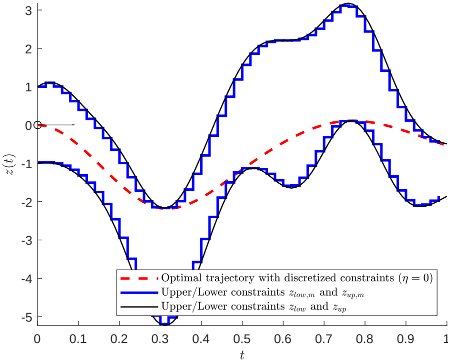
s.t.

$x_1(t_m) \in [z_{\text{low},m}, z_{\text{up},m}]$,

~~$\forall t \in [t_m - \delta_m, t_m + \delta_m]$~~, $\forall m \in [M]$



$$\bar{x}(\cdot) = \sum_{m=1}^{M} K_1(\cdot, t_m) p_m = \sum_{m=1}^{M} \alpha_m K_1(\cdot, t_m) e_m$$

$$K_1(s,t) = \int_0^{\min(s,t)} e^{(s-\tau)A} B B^\top e^{(t-\tau)A^\top} \mathrm{d}\tau$$

Numerical example: submarine in a cavern

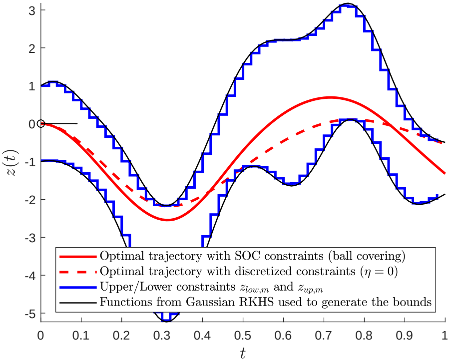
RKHS regression

$$\min_{x(\cdot)\in\mathcal{S}_u} \quad \|x(\cdot)\|_{K_1}^2$$

s.t.

$$x_1(t_m) \in [z_{\text{low},m}, z_{\text{up},m}] \pm \eta_m \|x(\cdot)\|_{K_1},$$

$$\sout{\forall t \in [t_m-\delta_m, t_m+\delta_m]}, \forall m \in [M]$$



$$\bar{x}(\cdot) = \sum_{m=1}^{M} K_1(\cdot, t_m)p_m = \sum_{m=1}^{M} \alpha_m K_1(\cdot, t_m)e_m$$

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Numerical example: submarine in a cavern

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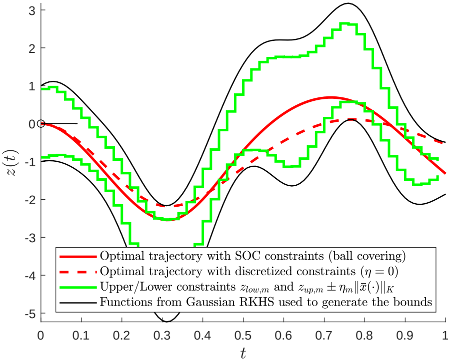
~~$\forall t \in [t_m - \delta_m, t_m + \delta_m]$~~, $\forall m \in [M]$



$$\bar{x}(\cdot) = \sum_{m=1}^{M} K_1(\cdot, t_m)p_m = \sum_{m=1}^{M} \alpha_m K_1(\cdot, t_m)e_m$$

$$K_1(s,t) = \int_0^{\min(s,t)} e^{(s-\tau)A}BB^\top e^{(t-\tau)A^\top}\,\mathrm{d}\tau$$

Numerical example 1: constrained pendulum - definition

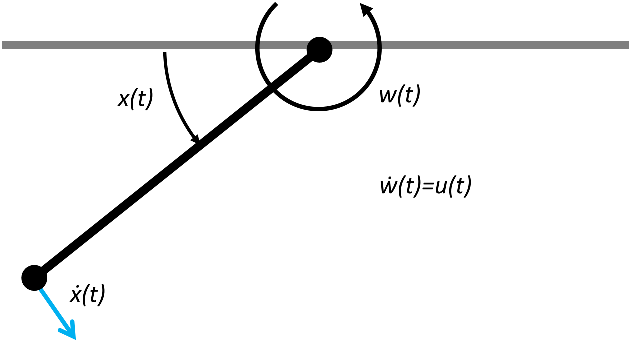
Constrained pendulum when controlling the third derivative of the angle

$$\min_{x(\cdot),w(\cdot),u(\cdot)} -\dot{x}(T) + \lambda \|u(\cdot)\|^2_{L^2(0,T)} \qquad \lambda \ll 1$$

$$x(0) = 0.5, \quad \dot{x}(0) = 0, \quad w(0) = 0, \quad x(T/3) = 0.5, \quad x(T) = 0$$

$$\ddot{x}(t) = -10\, x(t) + w(t), \quad \dot{w}(t) = u(t), \text{ a.e. in } [0, T]$$

$$\dot{x}(t) \in [-3, +\infty[, \quad w(t) \in [-10, 10], \forall\, t \in [0, T]$$



Numerical example 1: constrained pendulum - definition

Constrained pendulum when controlling the third derivative of the angle

$$\min_{x(\cdot),w(\cdot),u(\cdot)} -\dot{x}(T) + \lambda \|u(\cdot)\|^2_{L^2(0,T)} \qquad \lambda \ll 1$$

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$$\dot{x}(t) \in [-3, +\infty[, \quad w(t) \in [-10, 10], \forall\, t \in [0, T]$$

Converting affine state constraints to SOC constraints, applying rep. thm

$$\eta_{\dot{x}} \|x(\cdot)\|_K - \dot{x}(t_m) \le 3,$$
$$\eta_w \|x(\cdot)\|_K + w(t_m) \le 10,$$
$$\eta_w \|x(\cdot)\|_K - w(t_m) \le 10$$

$$\bar{x}(\cdot) = K(\cdot, 0)p_0 + K(\cdot, T/3)p_{T/3} + K(\cdot, T)p_T + \sum_{m=1}^{M} K(\cdot, t_m)p_m$$

Most of computational cost is related to the “controllability Gramians“ $K_1(s,t) = \int_0^{\min(s,t)} e^{(s-\tau)A} BB^\top e^{(t-\tau)A^\top} \mathrm{d}\tau$ which we have to approximate.

Numerical example 2: constrained pendulum - illustration

Optimal solutions of the constrained pendulum “path-planning“ problem. Red circles: equality constraints. Grayed areas: constraints over $[0, T]$.

Angle $x(\cdot)$ Velocity $\dot{x}(\cdot)$ Couple $w(\cdot)$

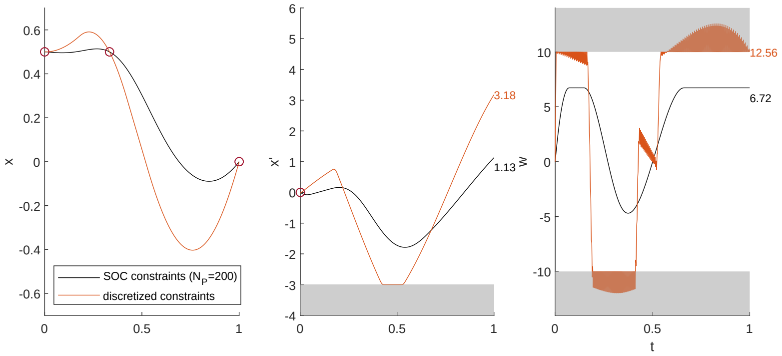

Figure: Comparison of SOC constraints (guaranteed η_w) vs discretized constraints ($\eta_w = 0$) for $N_P = 200$.

Numerical example 2: constrained pendulum - illustration

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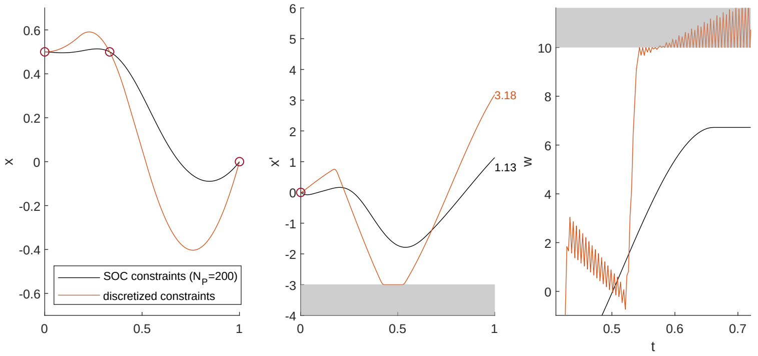

Figure: Comparison of SOC constraints (guaranteed η_w) vs discretized constraints ($\eta_w = 0$) for $N_P = 200$ - Chattering phenomenon like for traffic cameras!.

Optimal solutions of the constrained pendulum “path-planning“ problem.
Red circles: equality constraints. Grayed areas: constraints over $[0, T]$.

Angle $x(\cdot)$ Velocity $\dot{x}(\cdot)$ Couple $w(\cdot)$

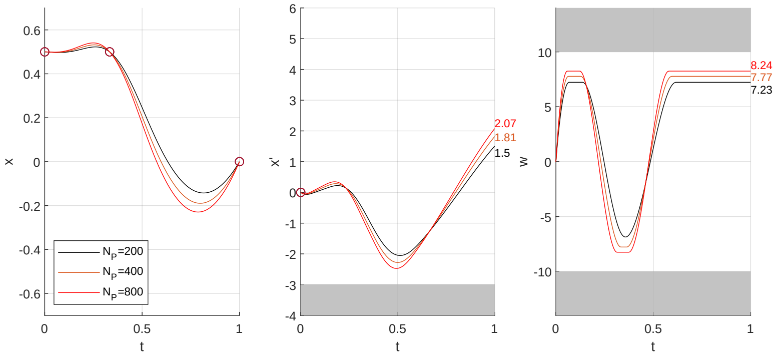

Figure: Comparison of SOC constraints for varying N_P and guaranteed η_w.

Numerical example 2: constrained pendulum - illustration

Optimal solutions of the constrained pendulum “path-planning“ problem.
Red circles: equality constraints. Grayed areas: constraints over $[0, T]$.

Angle $x(\cdot)$ Velocity $\dot{x}(\cdot)$ Couple $w(\cdot)$

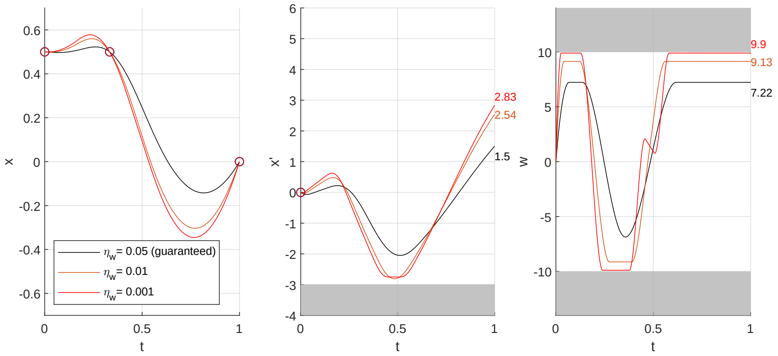

Figure: Comparison of SOC constraints for varying η_w and $N_P = 200$.

Pushing RKHSs beyond/Revisiting classical LQR

For RKHSs

- Control constraints do not correspond to continuous evaluations
  $\hookrightarrow$ limits of RKHS pointwise theory (e.g. $x' = u \in L^2([0, T], [-1, 1])$ a.e.)
- Successive linearizations of nonlinear system lead to changing kernels
  $\hookrightarrow$ a single kernel may not be sufficient (e.g. $x' = f_{[x_n(\cdot)]}x + f_{[u_n(\cdot)]}u$ a.e.)
- Non-quadratic costs for linear systems do not lead to Hilbert spaces
  $\hookrightarrow$ you may need Banach kernels (e.g. $\|u(\cdot)\|^2_{L^2(0,T)} \to \|u(\cdot)\|_{L^1(0,T)}$)

Pushing RKHSs beyond/Revisiting classical LQR

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- Control constraints do not correspond to continuous evaluations
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For control theory

- To each evaluation at time t corresponds a covector $p_t \in \mathbb{R}^N$
  $\hookrightarrow$ Representer theorem well adapted for state constraints, but unsuitable for control constraints. Reverts the difficulty w.r.t. PMP approach.

- The Gramian of controllability generates trajectories
  $\hookrightarrow$ This allows for close-form solutions in continuous-time.

Take-home messages

"Control problems can be seen as a form of machine learning with more constraints and less samples."

"The unconstrained LQR is a subset of kernel regression."

Contributions and Future problems

State-constrained LQR can be interpreted as a kernel regression which

- allows to revisit classical notions from the kernel viewpoint,
- allows to deal with the difficult problem of state constraints.

Open questions:

- From fixed $[t_0, T]$ to varying $t_0 \rightarrow$ Riccati equation
- Density of controlled trajectories in $\mathcal{C}^0 \rightarrow$ Controllability issues
- Operator-valued kernels $\rightarrow$ controlled PDE with state constraints

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Thank you for your attention!

References I

Aronszajn, N. (1950).
Theory of reproducing kernels.
Transactions of the American Mathematical Society, 68:337–404.

Aubin-Frankowski, P.-C. (2020).
Linearly-constrained Linear Quadratic Regulator from the viewpoint of kernel methods.
Technical report.
(https://arxiv.org/abs/2011.02196).

Berlinet, A. and Thomas-Agnan, C. (2004).
Reproducing Kernel Hilbert Spaces in Probability and Statistics.
Kluwer.

Heckman, N. (2012).
The theory and application of penalized methods or Reproducing Kernel Hilbert Spaces made easy.
Statistics Surveys, 6(0):113–141.

References II

Micheli, M. and Glaunès, J. A. (2014).
Matrix-valued kernels for shape deformation analysis.
Geometry, Imaging and Computing, 1(1):57–139.

Saitoh, S. and Sawano, Y. (2016).
Theory of Reproducing Kernels and Applications.
Springer Singapore.

Schölkopf, B., Herbrich, R., and Smola, A. J. (2001).
A generalized representer theorem.
In *Computational Learning Theory (CoLT)*, pages 416–426. Springer Berlin Heidelberg.

Van Loan, C. (1978).
Computing integrals involving the matrix exponential.
IEEE Transactions on Automatic Control, 23(3):395–404.

Annex: Van Loan's trick for time-invariant Gramians

Use matrix exponentials as in [Van Loan, 1978]

$$\exp\left(\begin{pmatrix} A & Q_c \\ 0 & -A^\top \end{pmatrix} \Delta\right) = \begin{pmatrix} F_2(\Delta) & G_2(\Delta) \\ 0 & F_3(\Delta) \end{pmatrix}$$

$$\hat{F}_2(t) = e^{At}$$

$$\hat{F}_3(t) = e^{-A^\top t}$$

$$\hat{G}_2(t) = \int_0^t e^{(t-\tau)A} Q_c e^{-\tau A^\top} \mathrm{d}\tau$$

$$K_1(s,t) = \int_0^{\min(s,t)} e^{(s-\tau)A} BB^\top e^{(t-\tau)A^\top} \mathrm{d}\tau$$

Set $Q_C = BR^{-1}B^\top$.

For $s \leq t,\ K_1(s,t) = \hat{G}_2(s)\hat{F}_2(t)^\top$

For $t \leq s,\ K_1(s,t) = \hat{F}_2(s)\hat{G}_2(t)^\top$

Annex: Green kernels and RKHSs

Let D be a differential operator, D^* its formal adjoint. Define the Green function $G_{D^*D,x}(y) : \Omega \to \mathbb{R}$ s.t. $D^*D\, G_{D^*D,x}(y) = \delta_x(y)$ then, if the integrals over the boundaries in Green's formula are null, for any $f \in \mathcal{F}_k$

$$f(x) = \int_\Omega f(y) D^* D G_{D^*D,x}(y) dy = \int_\Omega Df(y) DG_{D^*D,x}(y) =: \langle f, G_{D^*D,x} \rangle_{\mathcal{F}_k},$$

so $k(t,s) = G_{D^*D,x}(y)$ [Saitoh and Sawano, 2016, p61]. For vector-valued contexts, e.g. $\mathcal{F}_K = W^{s,2}(\mathbb{R}^d, \mathbb{R}^d)$ and $D^*D = (1 - \sigma^2 \Delta)^s$ component-wise, see [Micheli and Glaunès, 2014, p9].

Alternatively, in 1D, $D\, G_{D,x}(y) = \delta_x(y)$, the kernel associated to the inner product $\int_\Omega Df(y) Dg(y) dy$ for the space of f "null at the border" writes as

$$k(t,s) = \int_\Omega G_{D,x}(z) G_{D,y}(z) dz$$

see [Berlinet and Thomas-Agnan, 2004, p286] and [Heckman, 2012].

Annex: IPC gives strictly feasible trajectories

(H-sol) $C(0)x_0 < d(0)$ and there exists a trajectory $x^\epsilon(\cdot) \in \mathcal{S}$ satisfying strictly the affine constraints, as well as the initial condition.

(H1) $A(\cdot)$ and $B(\cdot)$ are $\mathcal{C}^0$. $C(\cdot)$ and $d(\cdot)$ are $\mathcal{C}^1$ and $C(0)x_0 < d(0)$.

(H2) There exists $M_u > 0$ s.t. , for all $t \in [0, T]$ and $x \in \mathbb{R}^N$ satisfying $C(t)x \leq d(t)$, and $\|x\| \leq (1 + \|x_0\|)e^{T\|A(\cdot)\|_{L^\infty(0,T)} + TM_u\|B(\cdot)\|_{L^\infty(0,T)}}$, there exists $u_{t,x} \in M_u\mathbb{B}_M$ such that

$$\forall\, i \in \{j \,|\, c_j(t)^\top x = d_j(t)\},\ c_i'(t)^\top x - d_i'(t) + c_i(t)^\top (A(t)x + B(t)u_{t,x}) < 0.$$

This is an inward-pointing condition (IPC) at the boundary.

Lemma (Existence of interior trajectories)

If (H1) and (H2) hold, then (H-sol) holds.

Annex: control proof main idea, nested property

$$\eta_i(\delta, t) := \sup \|K(\cdot, t)c_i(t) - K(\cdot, s)c_i(s)\|_K, \quad \omega_i(\delta, t) := \sup |d_i(t) - d_i(s)|,$$
$$d_i(\delta_m, t_m) := \inf d_i(s), \qquad \text{over } s \in [t_m - \delta_m, t_m + \delta_m] \cap [0, T]$$

For $\overrightarrow{\epsilon} \in \mathbb{R}_+^P$, the constraints we shall consider are defined as follows

$$\mathcal{V}_0 := \{x(\cdot) \in \mathcal{S} \mid C(t)x(t) \leq d(t), \forall\, t \in [0, T]\},$$
$$\mathcal{V}_{\delta,\text{fin}} := \{x(\cdot) \in \mathcal{S} \mid \overrightarrow{\eta}(\delta_m, t_m)\|x(\cdot)\|_K + C(t_m)x(t_m) \leq d(\delta_m, t_m), \forall\, m \in [\![1, N_0]\!]\},$$
$$\mathcal{V}_{\delta,\text{inf}} := \{x(\cdot) \in \mathcal{S} \mid \overrightarrow{\eta}(\delta, t)\|x(\cdot)\|_K + \overrightarrow{\omega}(\delta, t) + C(t)x(t) \leq d(t), \forall\, t \in [0, T]\},$$
$$\mathcal{V}_{\overrightarrow{\epsilon}} := \{x(\cdot) \in \mathcal{S} \mid \overrightarrow{\epsilon} + C(t)x(t) \leq d(t), \forall\, t \in [0, T]\}.$$

Proposition (Nested sequence)

Let $\delta_{max} := \max_{m \in [\![1,N_0]\!]} \delta_m$. *For any* $\delta \geq \delta_{max}$, *if, for a given* $y_0 \geq 0$, $\epsilon_i \geq \sup_{t \in [0,T]}[\eta_i(\delta, t)y_0 + \omega_i(\delta, t)]$, *then we have a nested sequence*

$$(\mathcal{V}_{\overrightarrow{\epsilon}} \cap y_0\mathbb{B}_K) \subset \mathcal{V}_{\delta,inf} \subset \mathcal{V}_{\delta,fin} \subset \mathcal{V}_0.$$

Only the simpler $\mathcal{V}_{\overrightarrow{\epsilon}}$ constraints matter!