Stability of solutions for controlled nonlinear systems under perturbation of state constraints

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Problem statement

- We consider a nonlinear system with unbounded control and state constraints

$$\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)), \qquad \text{for a.e. } t \in [0, T], \tag{1}$$

$$\mathbf{x}(t) \in \mathcal{A}_{0,t} := \{\mathbf{x} \,|\, \mathbf{h}(t, \mathbf{x}) \leq 0\}, \qquad \text{for all } t \in [0, T], \tag{2}$$

where $\mathbf{f} : [0, T] \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ and $\mathbf{h} : [0, T] \times \mathbb{R}^N \to \mathbb{R}^P$. We call **f**-trajectories the solutions of (1) for measurable controls $\mathbf{u}(\cdot)$.

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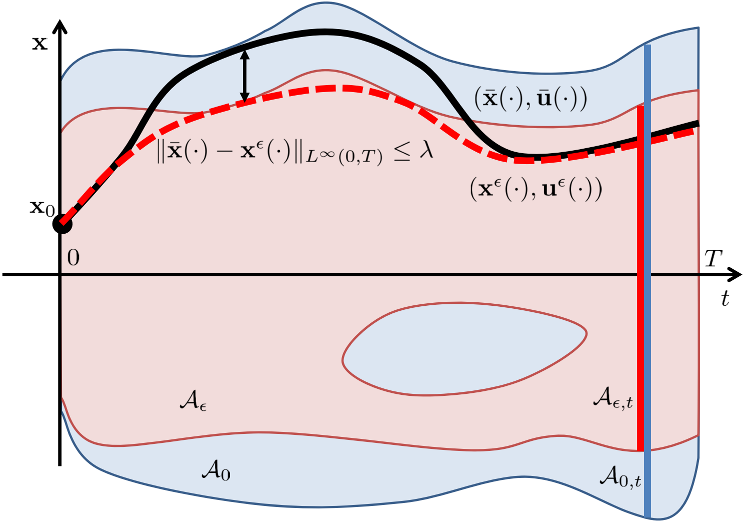
• We are given a reference trajectory $\bar{\mathbf{x}}(\cdot)$, such that $\mathbf{h}(0, \bar{\mathbf{x}}(0)) < 0$, with control $\bar{\mathbf{u}}(\cdot) \in L^\infty(0, T)$ satisfying (1)-(2). **Our goal** is to design a neighboring feasible trajectory $\mathbf{x}^\epsilon(\cdot)$ satisfying (1) and

$$\mathbf{x}^\epsilon(0) = \bar{\mathbf{x}}(0), \quad \|\bar{\mathbf{x}}(\cdot) - \mathbf{x}^\epsilon(\cdot)\|_{L^\infty(0,T)} \leq \lambda, \quad \left| \|\bar{\mathbf{u}}(\cdot)\|^2_{L^2(0,T)} - \|\mathbf{u}^\epsilon(\cdot)\|^2_{L^2(0,T)} \right| \leq \lambda, \quad \lambda \ll 1$$

$$\mathbf{x}^\epsilon(t) \in \mathcal{A}_{\epsilon,t} := \{\mathbf{x} \,|\, \epsilon + \mathbf{h}(t, \mathbf{x}) \leq 0\} \text{ for all } t \in [0, T]. \tag{3}$$

Important for interior point methods and perturbations!

Problem statement (illustrated)



Theorem

Under assumptions (H-1)-(H-6), for any $\lambda > 0$*, there exists* $\boldsymbol{\epsilon} > 0$ *and a* $\mathbf{f}$*-trajectory* $\mathbf{x}^{\epsilon}(\cdot)$ *on* $[0, T]$ *such that* $\mathbf{x}^{\epsilon}(0) = \bar{\mathbf{x}}(0)$*,* $\mathbf{x}^{\epsilon}(t) \in \operatorname{Int} \mathcal{A}_{\epsilon,t}$ *for all* $t \in [0, T]$*, and*

$$\|\bar{\mathbf{x}}(\cdot) - \mathbf{x}^{\epsilon}(\cdot)\|_{L^{\infty}(0,T)} \leq \lambda.$$

Moreover if (H-7) is satisfied, then, for any mapping $\mathbf{R}(\cdot) \in \mathcal{C}^0([0, T], \mathbb{R}^{M,M})$ *with positive semidefinite matrix values, one can choose* $\boldsymbol{\epsilon} > 0$ *and* $\mathbf{x}^{\epsilon}(\cdot)$ *such that the controls* $\mathbf{u}^{\epsilon}(\cdot)$ *satisfy*

$$\left| \|\mathbf{R}(\cdot)^{1/2}\bar{\mathbf{u}}(\cdot)\|^2_{L^2(0,T)} - \|\mathbf{R}(\cdot)^{1/2}\mathbf{u}^{\epsilon}(\cdot)\|^2_{L^2(0,T)} \right| \leq \lambda.$$

(Some) prior literature

$$F(t,x) = \{\mathbf{f}(t,\mathbf{x},\mathbf{u}) \,|\, \mathbf{u} \in U(t,\mathbf{x})\} \text{ and } \mathbf{x}(t) \in K(t) = \mathcal{A}_{0,t} := \{\mathbf{x} \,|\, \mathbf{h}(t,\mathbf{x}) \leq 0\}$$

bounded F: $\exists c \in \mathbb{R},\ F(t,x) \subset c(1+\|x\|)\mathbb{B}_N$ i.e. $\|\mathbf{f}(t,\mathbf{x},\mathbf{u})\| \leq c(1+\|x\|)$

- [Rampazzo, 1999]: bounded F and real-valued $h(\cdot) \in C^2$
- [Bettiol et al., 2010]: bounded F and real-valued $h(\cdot)$
- [Bressan and Facchi, 2011]: bounded F and compact convex K both time-independent + relaxation
- [Bettiol and Vinter, 2011]: bounded F and K time-independent + inf-relaxation

Control-affine systems $\mathbf{x}'(t) = \mathbf{a}(t,\mathbf{x}) + \mathbf{b}(t,\mathbf{x})\mathbf{u}$ with $\mathbf{u} \in \mathbb{R}^m$ are unbounded!!

Assumptions

$$\mathcal{A}_\epsilon := \{(t, \mathbf{x}) \,|\, t \in [0, T],\, \mathbf{x} \in \mathcal{A}_{\epsilon,t}\}.$$

(H-1) (Regular perturbation of $\mathcal{A}$)

$$\forall\, \lambda > 0,\ \exists\, \boldsymbol{\epsilon} > 0,\ \forall\, (t, \mathbf{x}) \in \mathcal{A}_0 \cap ([0, T] \times \|\bar{\mathbf{x}}(\cdot)\|_{L^\infty(0,T)} \mathbb{B}_N),$$
$$d_{\mathcal{A}_{\epsilon,t}}(\mathbf{x}) \leq \lambda.$$

(H-2) (Uniform continuity from the right of $d_{\partial \mathcal{A}_{\epsilon,t}}$ w.r.t. ϵ and t) There exist $\boldsymbol{\epsilon}_0 > 0$, $\Delta_0 > 0$, and $\omega_{\mathcal{A}}(\cdot) \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+)$ such that $\omega_{\mathcal{A}}(0) = 0$ and, for all $\boldsymbol{\epsilon} \leq \boldsymbol{\epsilon}_0$, and all $(t, \mathbf{x}) \in \mathcal{A}_0 \cap ([0, T] \times 2\|\bar{\mathbf{x}}(\cdot)\|_{L^\infty(0,T)} \mathbb{B}_N)$,

$$\forall\, \delta \in [0, \min(\Delta_0, T - t)],\ \|d_{\partial \mathcal{A}_{\epsilon,t+\delta}}(\mathbf{x}) - d_{\partial \mathcal{A}_{\epsilon,t}}(\mathbf{x})\| \leq \omega_{\mathcal{A}}(\delta).$$

Discussion: (H-1) and (H-2) are implied by $\mathcal{C}^{1,1}$-regularity of $\mathbf{h}$ with a surjective Jacobian $\frac{\partial \mathbf{h}(t,\mathbf{x})}{\partial \mathbf{x}}$ at all $(t, \mathbf{x}) \in \partial \mathcal{A}_0$.

Assumptions (cont'd)

$$\mathcal{A}_\epsilon := \{(t, \mathbf{x}) \,|\, t \in [0, T], \, \mathbf{x} \in \mathcal{A}_{\epsilon,t}\}.$$

(H-3) (Sublinear growth of $\mathbf{f}$ w.r.t. $\mathbf{x}$ and $\mathbf{u}$)

$$\exists\, \theta(\cdot) \in L^2_+(0, T), \; \forall\, t \in [0, T], \; \forall\, \mathbf{x} \in \mathbb{R}^N, \; \forall\, \mathbf{u} \in \mathbb{R}^M,$$
$$\|\mathbf{f}(t, \mathbf{x}, \mathbf{u})\| \leq \theta(t)(1 + \|\mathbf{x}\| + \|\mathbf{u}\|).$$

(H-4) (Inward-pointing condition) There exist $\epsilon_0 > 0$, $M_u > 0$, $M_v > 0$, $\xi > 0$, and $\eta > 0$ such that for all $\epsilon \leq \epsilon_0$ and all $(t, \mathbf{x}) \in (\partial\mathcal{A}_\epsilon + (0, \eta\mathbb{B}_N)) \cap \mathcal{A}_\epsilon \cap ([0, T] \times (1 + 2\|\bar{\mathbf{x}}(\cdot)\|_{L^\infty(0,T)})\mathbb{B}_N)$, we can find $\mathbf{u} \in M_u\mathbb{B}_M$ such that $\mathbf{v} := \mathbf{f}(t, \mathbf{x}, \mathbf{u})$ belongs to $M_v\mathbb{B}_N$ and

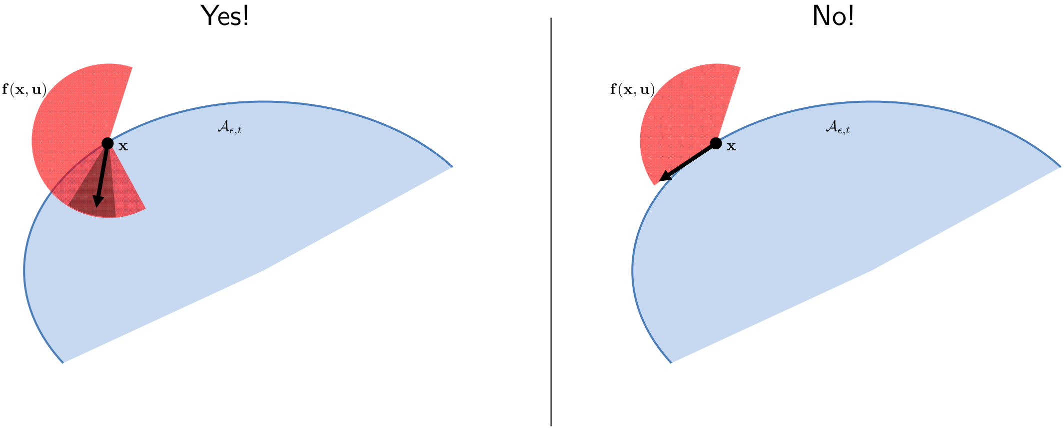
$$\mathbf{y} + \delta(\mathbf{v} + \xi\mathbb{B}_N) \subset \mathcal{A}_{\epsilon,t+\delta} \tag{4}$$

for all $\delta \in [0, \xi]$ and all $\mathbf{y} \in (\mathbf{x} + \xi\mathbb{B}_N) \cap \mathcal{A}_{\epsilon,t}$

$\hookrightarrow$ i.e. $F(t, x) \cap \mathrm{Int\,Tan}(\mathcal{A}_{\epsilon,t}(x)) \neq \emptyset$ on the boundary (Tan $=$ Clark tangent cone).

Discussion: (H-3) prevents finite-time explosion of trajectories. (H-4) is paramount to the construction (existence of control to "correct" the trajectory).

Inward pointing condition illustrated



We further ask the inward-pointing vector to be of bounded norm and bounded control, uniformly over $\partial\mathcal{A}_\epsilon$.

Assumptions (cont'd)

$$R := e^{\|\theta(\cdot)\|_{L^1(0,T)}} \Big[1 + \|\bar{\mathbf{x}}(\cdot)\|_{L^\infty(0,T)} + (1 + M_u)\|\theta(\cdot)\|_{L^1(0,T)} + \|\theta(\cdot)\|_{L^2(0,T)} (\|\bar{\mathbf{u}}(\cdot)\|_{L^2(0,T)} + \|\beta_u(\cdot)\|_{L^2(0,T)}) \Big]. \quad (5)$$

(H-6) (Local Lipschitz continuity of $\mathbf{f}$ w.r.t. $\mathbf{x}$)

$$\exists\, k_f(\cdot) \in L^2_+(0,T),\ \forall\, t \in [0,T],\ \forall\, \mathbf{x},\mathbf{y} \in R\mathbb{B}_N,\ \forall\, \mathbf{u} \in (M_u + \|\bar{\mathbf{u}}(\cdot)\|_{L^\infty(0,T)})\mathbb{B}_M,$$
$$\|\mathbf{f}(t,\mathbf{x},\mathbf{u}) - \mathbf{f}(t,\mathbf{y},\mathbf{u})\| \le k_f(t)\|\mathbf{x}-\mathbf{y}\|.$$

Discussion: (H-6) guarantees uniqueness and encompasses control-affine systems of the form $\mathbf{x}'(t) = \mathbf{a}(t,\mathbf{x}) + \mathbf{b}(t,\mathbf{x})\mathbf{u}$ with $\tilde{k}_f(t)$-Lipschitz functions $\mathbf{a}(t,\cdot)$ and $\mathbf{b}(t,\cdot)$, for some $\tilde{k}_f(\cdot) \in L^2(0,T)$.

Assumptions (cont'd)

(H-5) (Left local absolute continuity of $\mathbf{f}$ w.r.t. t)

$$\exists\, \gamma(\cdot) \in L^1_+(0, T),\ \exists\, \beta_u(\cdot) \in L^2_+(0, T),\ \forall\, 0 \le s < t \le T,\ \forall\, \mathbf{x} \in (1 + 2\|\bar{\mathbf{x}}(\cdot)\|_{L^\infty(0,T)})\mathbb{B}_N,$$
$$\forall\, \mathbf{u}_s \in (M_u + \|\bar{\mathbf{u}}(s)\|)\mathbb{B}_M,\ \exists\, \mathbf{u}_t \in \mathbf{u}_s + \beta_u(s)\mathbb{B}_M,$$
$$\|\mathbf{f}(t, \mathbf{x}, \mathbf{u}_t) - \mathbf{f}(s, \mathbf{x}, \mathbf{u}_s)\| \le \int_s^t \gamma(\sigma)\, d\sigma.$$

(H-7) (Hölderian selection of the controls in (H-5))

$$\exists\, \gamma(\cdot) \in L^1_+(0, T),\ \exists\, \alpha \in]0, 1],\ \exists\, k_u(\cdot) \in L^2_+(0, T),$$
$$\forall\, 0 \le s < t \le T,\ \forall\, \mathbf{x} \in (1 + 2\|\bar{\mathbf{x}}(\cdot)\|_{L^\infty(0,T)})\mathbb{B}_N,$$
$$\forall\, \mathbf{u}_s \in (M_u + \|\bar{\mathbf{u}}(s)\|)\mathbb{B}_M,\ \exists\, \mathbf{u}_t \in \mathbf{u}_s + (t - s)^\alpha k_u(s)\mathbb{B}_M,$$
$$\|\mathbf{f}(t, \mathbf{x}, \mathbf{u}_t) - \mathbf{f}(s, \mathbf{x}, \mathbf{u}_s)\| \le \int_s^t \gamma(\sigma)\, d\sigma.$$

Discussion: (H-5) was introduced to tackle discontinuities in the dynamics, and showcased on a civil engineering example [Bettiol et al., 2012, Section 4]. We adapt it to control systems and refine it in (H-7).

Idea of the proof

The overall strategy to construct a neighboring $\mathcal{A}_\epsilon$-feasible trajectory can be related to that of [Bettiol et al., 2012]. Modifying it to unbounded controls and time-varying constraints is however not straightforward.

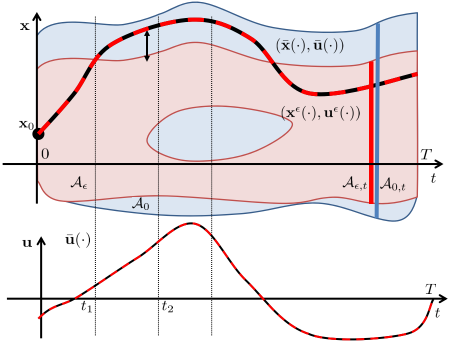
Consider small subintervals $[0, T] = \bigcup_{i \in [\![0, N_0-1]\!]} [t_i, t_{i+1}]$ and proceed iteratively.

- If the ith-trajectory stays in $\mathcal{A}_\epsilon$ over $[t_i, t_{i+1}]$, move to the next time interval.

- Otherwise, (H-4) provides us with an inward-pointing control $\mathbf{u}_i$ to stay in $\mathcal{A}_\epsilon$ for a short time.
  - apply $\mathbf{u}_i$ on $[t_i, t_i + t_\epsilon]$,
  - apply $\bar{\mathbf{u}}(\cdot - t_\epsilon)$ on $[t_i + t_\epsilon, t_{i+1}]$,
  - apply $\bar{\mathbf{u}}(\cdot)$ over $[t_{i+1}, T]$

By adequately choosing t_ϵ, we prove that the resulting control after N_0 iterations is L^2-close from $\bar{\mathbf{u}}(\cdot)$ and that the obtained trajectory is in $\mathcal{A}_\epsilon$.

Illustration of construction



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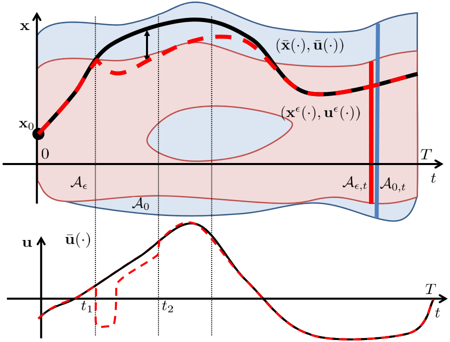

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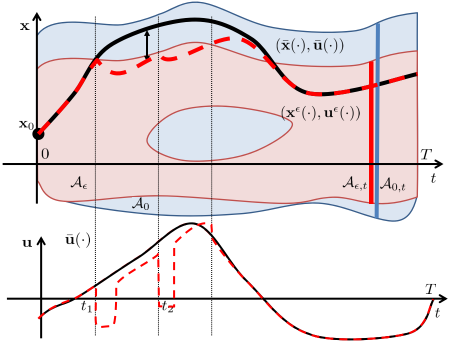

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Example: Consider an electric motor

$$x'(t) = a(t,x) + b(t,u),$$

with a bounded $a \in C^{1,1}([0,2] \times \mathbb{R}, \mathbb{R})$ and constraints $h(x) = 1 - |x|$, for controls $u \in \mathbb{R}$. The motor suffers an incident at $T = 1$. If it is a power surge

$$b(t,u) = \tilde{b}(t)u = \begin{cases} u & \text{if } t \in [0,1] \\ u/\sqrt[4]{t-1} & \text{if } t \in]1,2] \end{cases},$$

then (H-3) holds for $\theta \equiv \tilde{b} + \|f\|_\infty$ and so does (H-7) after some computation. If the incident consists in a power decline

$$b(t,u) = \begin{cases} \arctan(u) & \text{if } t \in [0,1] \\ (1 - \frac{\sqrt{t-1}}{2}) \arctan(u) & \text{if } t \in]1,2] \end{cases},$$

then the system is bounded and (H-7) holds with $u_t = u_s$, $\gamma(\sigma) = \frac{1}{4\sqrt{\sigma-1}}$ for $\sigma \in]1,2]$ and $\gamma(\sigma) = 0$ otherwise. In both cases (H-4) is satisfied, so perturbing the constraints still allows for a trajectory and control close to the reference ones as per Theorem 1.

Conclusion & Extensions

- We have proven that one can approximate trajectories of systems with unbounded control (e.g. control-affine) under assumptions similar to those of bounded systems.

- Systems $\tilde{\mathbf{f}}$ with Lipschitz (or Hölderian) control constraints $t \rightsquigarrow U(t)$ can be considered by projecting over $U(t)$, i.e. $\mathbf{f}(t, \mathbf{x}, \mathbf{u}) = \tilde{\mathbf{f}}(t, \mathbf{x}, \mathrm{proj}_{U(t)}(\mathbf{u}))$ if $\mathbf{f}$ satisfies the above assumptions.

- State constraints of order 2 (or more), e.g. $\ddot{x} = u$ with x constrained, do not enter into the proposed framework (requires Lie brackets, see Franco Rampazzo's recent work)

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